UNIFORM LOWER TRIANGULAR MATRIX SUMMABILITY OF A FOURIER SERIES

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Abstract. In this paper, the concept of uniform triangular matrix summability has been introduced and a new theorem on uniform lower triangular matrix summability has been established so that this theorem generalizes all the works of this direction.

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1. DEFINITIONS AND NOTATIONS

Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $[-\pi,\pi]$.The Fourier series associated with this function is

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

where a_0, a_n, b_n are known as Fourier trigonometric coefficients of $f(x)$ and are given by :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad n = 1,2,3\ldots. \tag{1.2}$$

Let $\sum_{n=0}^{\infty} u_n(x)$ be an infinite series defined in $[a,b] \subset [-\pi,\pi]$. The n^{th} partial sum of the series $\sum_{n=0}^{\infty} u_n(x)$ is given by $S_n(x) = \sum_{\nu=0}^{n} u_\nu(x)\ \forall\ x \in [a,b]$.

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying Silverman-Töeplitz [6] conditions of regularity i.e.

$$\text{(i)} \sum_{k=0}^{n} a_{n,k} \to 1 \quad as \quad n \to \infty$$

$$\text{(ii)} \; a_{n,k} = 0 \quad for \quad k > n$$

and $$\text{(iii)} \sum_{k=0}^{n} \left| a_{n,k} \right| \leq M$$ where M is finite constant.

If there exists a bounded function $S(x)$ such that

$$t_n(x) = \sum_{k=0}^{n} a_{n,k} \{ S_k(x) - S(x) \}$$
$$= o(1) \quad as \quad n \to \infty$$

uniformly $\forall\, x \in [a,b]$ then we say that the series $\sum_{n=0}^{\infty} u_n(x)$ is summable (T) uniformly in $a \leq x \leq b$ to the sum $S(x)$.

Particular Cases. The important particular cases of the triangular matrix means are:

(*i*) Cesàro mean of order 1 or (*C*, 1) mean if $a_{n,k} = \dfrac{1}{n+1} \; \forall\, k$.

(*ii*) Harmonic means when $a_{n,k} = \dfrac{1}{(n-k+1)\log n}$.

(*iii*)(*C*, δ) means when $a_{n,k} = \dfrac{\binom{n-k+\delta-1}{\delta-1}}{\binom{n+\delta}{\delta}}$.

(*iv*)(*H*, *p*) means when $a_{n,k} = \dfrac{1}{(\log)^{p-1}(n+1)} \prod_{q=0}^{p-1} \log^{q}(k+1)$.

(*v*) Nörlund means [1919] when $a_{n,k} = \dfrac{p_{n-k}}{P_n}$ where $P_n = \sum_{k=0}^{\infty} p_k$, $P_n \neq 0$.

(*vi*) Riesz means $(\overline{N}, p_n)$ when $a_{n,k} = \dfrac{p_k}{P_n}$, $P_n \neq 0$.

(*vii*)Generalised Nörlund Means (*N*, *p*, *q*) when $a_{n,k} = \dfrac{p_{n-k} q_k}{R_n}$.

where $R_n = \sum_{k=0}^{\infty} p_k q_{n-k}$, $R_n \neq 0$.

We write
$$\phi(t) = f(x+t) + f(x-t) - 2S(x), \tag{1.3}$$

$$\Phi(t) = \int_0^t |\phi(u)| du \tag{1.4}$$

$$A_{n,\tau} = \sum_{k=o}^{\tau} a_{n,n-\tau} = \sum_{k=n-\tau}^{n} a_{n,k} , \tag{1.5}$$

where $$\tau = \left[\frac{1}{t}\right] = \text{integral part of } \frac{1}{t}, \tag{1.6}$$

and $$K_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}} . \tag{1.7}$$

2. INTRODUCTION

Siddiqi [5] proved the following theorem:

TheoremA. If

$$\Phi(t) = o\left[\frac{t}{\log(1/t)}\right] \tag{2.1}$$

as $t \to +0$, then the series (1.1), at $t = x$ is summable (*H*) to $f(x)$.

Singh [8] generalized the above theorem for (N, p_n) summability in the following form:

TheoremB. Under the condition (2.1), the Fourier series of $f(t)$, at $t = x$, is summable (N, p_n) to $f(x)$, where $\{p_n\}$ is non-negative, non-increasing sequence such that

$$\sum_{k=\alpha}^{n} \frac{P_k}{k \log k} = O(P_n),$$

where $\alpha > 1$ is a fixed positive integer.

Continuing the study for (N, p_n) summability, Pati [7] has proved the following therem:

TheoremC. If (N, p_n) be a regular Nörlund method, defined by a real, non-negative, monotonic, non-increasing sequence of the coefficient $\{p_n\}$ such that $P_n \to \infty$, and $\log n = O(P_n)$ as $n \to \infty$ then if

$$\Phi(t) = \int_0^t \phi(t) dt = o\left[\frac{t}{P_\tau}\right] \tag{2.2}$$

as $t \to +0$, the Fourier series of $f(t)$, at $t = x$ is summable (N, p_n) to $f(x)$.

Dealing with uniform summability method, Saxena [2] established the following theorem:

TheoremD: If $$\Phi(t) = o\left[\frac{t}{\log(1/t)}\right],$$

uniformly in a set E in which $S = S(x)$ is bounded, as $t \to +0$, then the series (1.1) is summable by Harmonic means uniformly in E to the sum S.

Saxena [3] generalizes above theorem for uniform Nörlund summability method in the following form:

TheoremE: If $\alpha(t)$ stands for a function of t and $\alpha(t)$ ultimately increase steadily with t,

$$\int_{\frac{1}{n}}^{\delta} \frac{P_\tau}{\alpha(P_\tau)} \cdot \frac{1}{t} dt = O(P_n), \text{ as } n \to \infty, \tag{2.3}$$

and
$$\Phi(t) = o\left(\frac{t}{\alpha(P_\tau)}\right), \tag{2.4}$$

uniformly in E in which $S = S(x)$ is bounded, as $t \to +0$, then the series (1.1) is summable (N, p_n) uniformly in E to the sum S.

3. MAIN THEOREM.

Quite a good amount of works are known for uniform harmonic as well as Nörlund summability of Fourier series. In this paper, a more general result than those of Siddiqi [5], Saxena [2, 3], Pati [7], and Singh [8] has been established so that their results come out as particular cases.

Theorem. Let $T = (a_{n,k})$ be an infinite triangular matrix such that the elements $(a_{n,k})$ are non-negative and non-decreasing with $k \le n$ such that $A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-\tau} = \sum_{k=n-\tau}^{n} a_{n,k}$, $A_{n,n} = 1 \ \forall\, n$. If

$$\int_0^t |\phi(u)|\, du = o\left(\frac{t}{\xi(1/t)\log(1/t)}\right), \tag{3.1}$$

uniformly in a set E = $[a,b]$ in which $S(x)$ is bounded, as $t \to +0$, where $\xi(t)$ is a positive, monotonic increasing function of t such that

$$\int_{\frac{1}{\delta}}^{n} \frac{A_{n,u}\, du}{u\,\xi(u)\log u} = O(1), \tag{3.2}$$

as $n \to \infty$, for $0 < \delta < 1$,then the Fourier series (1.1) is lower matrix summable (T) uniformly in E = $[a,b]$ to the sum $S(x)$.

4. LEMMAS.

We shall require the following lemmas for the proof of our theorem-

Lemma4.1. Let $K_n(t)$ be given by (1.7) then $K_n(t) = O(n)$, $0 < t \le \frac{1}{n}$.

Proof: $$K_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}$$

$$|K_n(t)| = \frac{1}{2\pi}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2\pi}\sum_{k=0}^{n}|a_{n,k}|.\left|\frac{\sin(2k+1)\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2\pi}\sum_{k=0}^{n}|a_{n,k}|.\frac{(2k+1)|\sin\frac{t}{2}|}{|\sin\frac{t}{2}|}$$

$$\le \frac{(2n+1)}{2\pi}\sum_{k=0}^{n}|a_{n,k}|$$

$$\le \frac{(n+1)}{\pi}.M \quad \text{by Töeplitz [6] condition of regularity}$$

$$= O(n).$$

Lemma.4.2. If $a_{n,k}$ is a non-negative and non-decreasing with *k*, then

$$\left|\sum_{k=0}^{n} a_{n,k}\sin(k+\tfrac{1}{2})t\right| = O(A_{n,\tau}) \quad \text{for} \quad 0 < \frac{1}{n} \le t < \delta < \pi .$$

Proof: $$\left|\sum_{k=0}^{n} a_{n,k}\sin(k+\tfrac{1}{2})t\right| \le \left|\sum_{k=0}^{n-\tau} a_{n,k}\sin(k+\tfrac{1}{2})t\right| + \left|\sum_{k=n-\tau}^{n} a_{n,k}\sin(k+\tfrac{1}{2})t\right|$$

$$\le 2a_{n,n-\tau}\max_{0\le k\le r\le n-\tau}\left|\sum_{k=0}^{r}\sin(k+\tfrac{1}{2})t\right| + \sum_{k=n-\tau}^{n} a_{n,k}|\sin(k+\tfrac{1}{2})t|,$$

(by Abel's Lemma)

$$\le 2a_{n,n-\tau}\left|\frac{\sin^2(r+1)\frac{t}{2}}{\sin\frac{t}{2}}\right| + A_{n,\tau}$$

$$\left|\sum_{k=0}^{n} a_{n,k}\sin(k+\tfrac{1}{2})t\right| \le \frac{2a_{n,n-\tau}}{t} + A_{n,\tau} \tag{4.1}$$

Now $$A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k} = \sum_{k=n-\tau}^{n} a_{n,k}$$

$$= a_{n,n-\tau} + a_{n,n-\tau+1} + ...+ a_{n,n}$$

$$\ge (\tau+1)a_{n,n-\tau}$$

$$\ge \frac{a_{n,n-\tau}}{t} \qquad (\text{ since } \tau = \lfloor \tfrac{1}{t} \rfloor) .$$

Therefore $\frac{a_{n,n-\tau}}{t} = O(A_{n,\tau})$. (4.2)

By (4.1) and (4.2), we have $\left|\sum_{k=0}^{n} a_{n,k} \sin(k+\tfrac{1}{2})t\right| = O(A_{n,\tau})$.

Lemma.4.3. If $a_{n,k}$ is non-negative and non-decreasing with $k \le n$ and $K_n(t)$ is given by (1.7) then $K_n(t) = O\left(\frac{A_{n,\tau}}{t}\right)$ for $0 < \frac{1}{n} \le t < \delta < \pi$.

Proof: Since for $0 < \frac{1}{n} \le t < \delta < \pi$, $\sin t \ge \frac{t}{\pi}$,

We have $|K_n(t)| = \frac{1}{2\pi}\left|\sum_{k=0}^{n} a_{n,k} \frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right|$

$$\le \frac{1}{2\pi \sin\frac{t}{2}}\left[\left|\sum_{k=0}^{n} a_{n,k} \sin(k+\tfrac{1}{2})t\right|\right]$$

$$\le \frac{1}{2\pi}.\frac{2\pi}{t}\left[O(A_{n,\tau})\right] \quad \text{from lemma (4.2)}$$

$$|K_n(t)| = O\left(\frac{A_{n,\tau}}{t}\right).$$

Hence the lemma is proved.

5. PROOF OF THE MAIN THEOREM.

Following Titchmarsh [4], we have –

$$S_k(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}.\phi(t)dt \quad \text{uniformly in } a \le x \le b.$$

Then $t_n(x) = \sum_{k=0}^{n} a_{n,k}\{S_k(x) - f(x)\}$

$$= \frac{1}{2\pi}\int_0^{\pi}\left(\sum_{k=0}^{n} a_{n,k}.\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right).\phi(t)dt$$

$$= \int_0^{\pi} K_n(t).\phi(t)dt$$

$$= \int_0^{\frac{1}{n}} K_n(t).\phi(t)dt + \int_{\frac{1}{n}}^{\delta} K_n(t).\phi(t)dt + \int_{\delta}^{\pi} K_n(t).\phi(t)dt$$

$$= I_1 + I_2 + I_3 \quad \text{uniformly in } a \le x \le b.$$

By Riemann Lebesgue theorem and regularity conditions we get $I_3 = o(1)$.

And now $I_1 = \int_0^{\frac{1}{n}} K_n(t).\phi(t)dt$

$$\leq \int_0^{\frac{1}{n}} |K_n(t)|.|\phi(t)|dt$$

$$= O(n).\int_0^{\frac{1}{n}} |\phi(t)|dt \quad \text{by lemma (4.1)}$$

$$= O(n).o\left(\frac{1}{n\xi(n)\log n}\right) \text{, by condition (3.1).}$$

$$= o\left(\frac{1}{\xi(n)\log n}\right)$$

$$= o(1) \text{ as } n \to \infty.$$

Now $I_2 = \int_{\frac{1}{n}}^{\delta} K_n(t).\phi(t)dt$,

$$|I_2| \leq \int_0^{\frac{1}{n}} |K_n(t)|.|\phi(t)|dt$$

$$= O(1).\int_{\frac{1}{n}}^{\delta} \left(\frac{A_{n,\tau}}{t}\right).|\phi(t)|dt$$

$$= O(1).\left[\left[\frac{A_{n,\tau}}{t}.\Phi(t)\right]_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta} \frac{d}{dt}\left(\frac{A_{n,\tau}}{t}\right).\Phi(t)dt\right]$$

$$\leq O(1).\left[\left[\frac{A_{n,\tau}}{t}.o\left(\frac{t}{\xi(\frac{1}{t})\log(\frac{1}{t})}\right)\right]_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta}\left(\frac{A_{n,\tau}}{t^2}\right).o\left(\frac{t}{\xi(\frac{1}{t})\log(\frac{1}{t})}\right)dt + \int_{\frac{1}{n}}^{\delta}\frac{1}{t}\left[\frac{d}{dt}(A_{n,\tau})\right].o\left(\frac{t}{\xi(\frac{1}{t})\log(\frac{1}{t})}\right)dt\right]$$

$$\leq o(1)\left[\frac{A_{n,[\frac{1}{\delta}]}}{\xi(\frac{1}{\delta})\log(\frac{1}{\delta})} + \frac{A_{n,n}}{\xi(n)\log n} + \int_{\frac{1}{n}}^{\delta}\frac{A_{n,\tau}dt}{t\xi(\frac{1}{t})\log(\frac{1}{t})} + \int_{\frac{1}{n}}^{\delta}\frac{1}{\xi(\frac{1}{t})\log(\frac{1}{t})}.d(A_{n,\tau})\right]$$

$$= o(1) + o(1).\int_{\frac{1}{n}}^{\delta}\frac{A_{n,\tau}dt}{t\xi(\frac{1}{t})\log(\frac{1}{t})} + o(1).\int_{\frac{1}{n}}^{\delta}\frac{1}{\xi(\frac{1}{t})\log(\frac{1}{t})}.d(A_{n,\tau})$$

$$= o(1) + o(1).\int_{\frac{1}{n}}^{\delta}\frac{A_{n,\tau}dt}{t\xi(\frac{1}{t})\log(\frac{1}{t})} + o(1).\int_{\frac{1}{\delta}}^{n}\frac{d(A_{n,u})}{\xi(u)\log u}$$

$$= o(1) + o(1).\int_{\frac{1}{\delta}}^{n}\frac{A_{n,u}du}{u\xi(u)\log(u)} + o\left(\frac{1}{\xi(\frac{1}{\delta})\log(\frac{1}{\delta})}\sum_{k=[\frac{1}{\delta}]+1}^{n} a_{n,k}\right) + o\left(\frac{1}{\xi(n)\log(n)}\sum_{k=[\frac{1}{\delta}]+1}^{n} a_{n,k}\right)$$

by mean value theorem for integrals

$= o(1)$ as $n \to \infty$, by condition (3.2)

which completes the proof of the main theorem.

Particular cases.(a) If $a_{n,k} = \dfrac{1}{(n-k+1)\log n}$, $\xi(t) = 1 \ \forall t$, $[a,b] = \{x\}$ then the result of Siddiqi [5] becomes a particular case of our theorem.

(b) The result of Singh[8] is a particular case of our theorem if $a_{n,k} = \dfrac{p_{n-k}}{P_n}, P_n = \sum_{k=0}^{n} p_k$ and $[a,b] = \{x\}$, $\xi(t) = 1 \ \forall t$

(c) If $a_{n,k}$ is defined as in case (b), $[a,b] = \{x\}$ and $\xi(t) = \dfrac{P_{[t]}}{\log t}$ then our theorem reduces to theoremC by Pati [7].

(d) If $a_{n,k}$ and $\xi(t)$ is defined as in case (a) and $[a,b] =$ set E, then the result of Saxena [2] is a particular case of our theorem.The condition of Saxena [2] is analogous to the result of Siddiqi [5].

(e) If $a_{n,k}$ is defined as in case (b) and $\xi(t) = \dfrac{\alpha(P_{[t]})}{\log t}$, $[a,b] =$ set E, then the result of Saxena [3] is a particular case of our theorem.

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