A Note on the form of Jacobi Polynomial used in Harish-Chandra's Paper 'Motion of an Electron in the Field of a Magnetic Pole'

Vinod Kumar Yadav

Junior Research Fellow (CSIR), Department of Mathematics,
V.S.S.D. (P.G.) College, Kanpur-208002, U.P. (INDIA)
E-mail: vinodkumaryadav3@gmail.com

Abstract: It is very interesting and difficult to understand the papers of the great mathematician Prof. Harish-Chandra (1923-1983). While reading any one of them the reader is compelled to know the answers of many questions standing on the way. We have tried to understand his paper[1] and found that the Jacobi polynomial appears on the way of solution of wave equation of electron moving in the field of a magnetic pole. This Jacobi polynomial is not in the usual form appearing in mathematical literature. In this paper we have compared the Jacobi polynomial used by Harish-Chandra with its usual form. We have also deduced the explicit form for such polynomials from identity given in his paper[1] . We have also verified the results found by him concerning Jacobi polynomials.

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1. INTRODUCTION

In his paper[1] Prof. Harish-Chandra obtained the suitable Hamiltonian H for the motion of electron in the field of a magnetic pole and reduced the problem to find the wave function ψ satisfying the wave equation

(1.1) $$H\psi = E\psi ,$$

where E is some eigenvalue of H. The spherical polar coordinate system is suitable for the problem and therefore using the transformation laws of tensor analysis he converted (1.1) to the following form

(1.2) $$\left[\frac{1}{i}\rho_1\left\{\sigma_3\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left\{M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3}{2}\cos\theta\right\}\right)\right\}+\rho_3\mu+E\right]\psi_0=0 .$$

Where he has written $\psi = e^{i\left(M-\frac{1}{2}\sigma_3\right)\phi-\frac{1}{2}i\sigma_2\theta}\psi_0$, to make the equation free from ϕ, i.e. ψ_0 is a function of r, θ only and μ is mass of electron and M is half an odd integer. The two independent sets of Pauli operators[2] σ 's and ρ 's satisfy the relations

(1.3) $$\begin{cases}\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}\,,\ \rho_i\rho_j+\rho_j\rho_i=2\delta_{ij}\\ \sigma_i\rho_j=\rho_j\sigma_i\,,\ \sigma_1\sigma_2\sigma_3=i\,,\ \rho_1\rho_2\rho_3=i\end{cases}\qquad (i,j=1,2,3),$$

and commute with all other operators involved in the equation (1.2).

As in paper[1] Harish-Chandra assumed that

(1.4) $$-K^2=\left\{\sigma_1\left[\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3\cos\theta}{2}\right)\right]\right\}^2,$$

because the operator within the curly bracket is purely imaginary. This operator commutes with the operator acting on ψ_0 in (1.2). This fact can be verified in the following way. Being the multiple of the operator in the curly bracket in (1.4) by $\frac{1}{i}\frac{\rho_1}{r}$, the operator $\frac{1}{i}\rho_1\left\{\frac{\sigma_1}{r}\left(\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left\{M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3}{2}\cos\theta\right\}\right)\right\}$ commutes with $-K^2$. Further $-K^2$ clearly commutes with $\rho_3\mu+E$, since ρ_3 commutes with σ_1 and σ_3. By bringing σ_1 to the left of first term of the square we may write $-K^2$ as $\left\{\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta+\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}\left\{\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}$,this makes easier to understood that it commutes with $\frac{1}{i}\rho_1\left(\sigma_3\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\right)$. Thus we find that $-K^2$ commutes with the operator of (1.2). Hence $-K^2$ equal to the square of operator

$$\sigma_1\left[\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3\cos\theta}{2}\right)\right].$$

Since if two operators commute their eigenvectors are same though their eigenvalues may be different. Hence we can choose ψ_0 to be an eigenvector of $-K^2$.

2. REDUCTION OF THE OPERATOR $-K^2$ TO THE JACOBI OPERATOR

From equation (1.4)

$$-K^2=\left\{\sigma_1\left[\frac{\partial}{\partial\theta}-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)-\frac{\sigma_3\cos\theta}{2}\right)\right]\right\}^2$$

$$=\left\{\sigma_1\left[\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right]\sigma_1\right\}\times$$
$$\left\{\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}$$

$$=\left\{\sigma_1^2\left[\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta+\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right]\right\}\times$$
$$\left\{\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta-\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}$$

$$=\left\{\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta+\frac{\sigma_3}{\sin\theta}\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}\times$$

$$\frac{1}{\sin\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{2}\cos\theta-\sigma_3\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\},$$

Since $\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}-\frac{\cot\theta}{\sin\theta}$ we get

$$(2.1)\qquad -K^2=\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}-\frac{1}{2}\cot\theta+\sigma_3\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}\times$$
$$\left\{\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{2}\cos\theta-\sigma_3\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}.$$

Now for finding eigenvectors of $-K^2$, he puts $u=\cos\theta$, $\sin\theta=\sqrt{1-u^2}$ and $\frac{\partial}{\partial\theta}=\frac{\partial u}{\partial\theta}\frac{\partial}{\partial u}=-\sqrt{1-u^2}\frac{\partial}{\partial u}$.

∴ First factor of R.H.S. of equation (2.1)

$$=\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}-\frac{1}{2}\cot\theta+\sigma_3\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}$$
$$=\frac{1}{\sqrt{1-u^2}}\left\{-\sqrt{1-u^2}\frac{d}{du}-\frac{1}{2}\frac{u}{\sqrt{1-u^2}}+\frac{\sigma_3}{\sqrt{1-u^2}}\left(M+\frac{n}{2}(1-u)\right)\right\}$$
$$=-\frac{1}{1-u^2}\left\{\left(1-u^2\right)\frac{d}{du}-\sigma_3\left(M+\frac{n}{2}\right)+\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\right\},$$

Also second factor of R.H.S. of equation (2.1)

$$=\left\{\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{2}\cos\theta-\sigma_3\left(M+\frac{n}{2}(1-\cos\theta)\right)\right\}$$
$$=\left\{\sqrt{1-u^2}\left(-\sqrt{1-u^2}\frac{d}{du}\right)+\frac{1}{2}u-\sigma_3\left(M+\frac{n}{2}(1-u)\right)\right\}$$
$$=-\left\{\left(1-u^2\right)\frac{d}{du}+\sigma_3\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\right\}.$$

Now equation (2.1) becomes

$$-K^2=-\frac{1}{1-u^2}\left\{\left(1-u^2\right)\frac{d}{du}-\sigma_3\left(M+\frac{n}{2}\right)+\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\right\}\times$$
$$(-1)\left\{\left(1-u^2\right)\frac{d}{du}+\sigma_3\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\right\}$$
$$=\frac{1}{1-u^2}\left\{\left(1-u^2\right)\left[\left(1-u^2\right)\frac{d^2}{du^2}+(-2u)\frac{d}{du}+\sigma_3\left(M+\frac{n}{2}\right)\frac{d}{du}-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)\right.\right.$$

$$-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\frac{d}{du}\Bigg]-\sigma_3\left(M+\frac{n}{2}\right)\left(1-u^2\right)\frac{d}{du}-\sigma_3^{\,2}\left(M+\frac{n}{2}\right)^2$$
$$+\sigma_3\left(M+\frac{n}{2}\right)\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u+\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\left(1-u^2\right)\frac{d}{du}$$
$$+\sigma_3\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)\left(M+\frac{n}{2}\right)u-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)^2u^2\Bigg\}$$

$$=\left(1-u^2\right)\frac{d^2}{du^2}-2u\frac{d}{du}-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)$$
$$-\frac{\left\{\sigma_3^{\,2}\left(M+\frac{n}{2}\right)^2-2\sigma_3\left(M+\frac{n}{2}\right)\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u+\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)^2u^2\right\}}{\left(1-u^2\right)}$$

$$=\left(1-u^2\right)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\left\{\sigma_3\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)u\right\}^2}{\left(1-u^2\right)}-\left(\frac{n}{2}\sigma_3+\frac{1}{2}\right)$$

$$=\left(1-u^2\right)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\sigma_3^{\,2}\left\{\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}+\frac{\sigma_3}{2}\right)u\right\}^2}{\left(1-u^2\right)}-\sigma_3^{\,2}\left(\frac{n}{2}+\frac{\sigma_3}{2}\right)^2+\frac{n^2}{4}-\frac{1}{4}$$

$$(2.2)\quad -K^2=\left(1-u^2\right)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\left\{\left(M+\frac{n}{2}\right)-\left(\frac{n}{2}+\frac{\sigma_3}{2}\right)u\right\}^2}{\left(1-u^2\right)}-\left(\frac{n}{2}+\frac{\sigma_3}{2}\right)^2+\frac{n^2-1}{4},$$

where $u=\cos\theta$. This is the required form of operator $-K^2$.

3. EIGENVALUES AND EIGENFUNCTIONS OF OPERATOR $-K^2$

In paper[1] Prof. Harish-Chandra put $m=\left(M+\frac{n}{2}\right)$, $j=\left(\frac{n}{2}+\frac{\sigma_3}{2}\right)$ and say that if m, j are both integral or both half-integral the only eigenfunctions of the operator

$$(3.1)\qquad \left(1-u^2\right)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\{m-ju\}^2}{\left(1-u^2\right)}-j^2,$$

corresponding to the interval $-1\le u\le 1$ are the Jacobi polynomials $P^k_{m,j}(u)$ and corresponding eigenvalues are $-k(k+1)$, where k is to be so chosen that $k\ge|m|$, $|j|$ and

$k-j$ is an integer. Prof. Harish-Chandra defined $P_{m,j}^{k}(\cos\theta)$ by nice and beautiful identity, which is one of the achievements of paper[1] is given as

(3.2) $$\frac{\left(t_1\cos\frac{\theta}{2}+t_2\sin\frac{\theta}{2}\right)^{k-j}\left(-t_1\sin\frac{\theta}{2}+t_2\cos\frac{\theta}{2}\right)^{k+j}}{\left\{(k-j)!(k+j)!\right\}^{\frac{1}{2}}}$$
$$=\sum_{m=k}^{-k}\frac{t_1^{\,k-m}t_2^{\,k+m}}{\left\{(k-m)!(k+m)!\right\}^{\frac{1}{2}}}P_{m,j}^{k}(\cos\theta).$$

Thus from (3.1) we get

(3.3) $$\left\{\left(1-u^2\right)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\{m-ju\}^2}{\left(1-u^2\right)}-j^2\right\}P_{m,j}^{k}(u)=-k(k+1)P_{m,j}^{k}(u)$$

i.e. $$\left\{\left(1-u^2\right)\frac{d^2}{du^2}-2u\frac{d}{du}-\frac{\{m-ju\}^2}{\left(1-u^2\right)}-j^2+\frac{n^2-1}{4}\right\}P_{m,j}^{k}(u)$$
$$=\left\{-k(k+1)+\frac{n^2-1}{4}\right\}P_{m,j}^{k}(u)$$

i.e. $$-K^2\,P_{m,j}^{k}(u)=-\left\{\left(k+\frac{n+1}{2}\right)\left(k-\frac{n-1}{2}\right)\right\}P_{m,j}^{k}(u).$$

So eigenfunctions of the operator $-K^2$ in the interval $-1\le u\le 1$ are the Jacobi polynomials $P_{m,j}^{k}(u)$ and the corresponding eigenvalues are $-\left\{\left(k+\frac{n+1}{2}\right)\left(k-\frac{n-1}{2}\right)\right\}$.

Also $$K^2\,P_{m,j}^{k}(u)=\left\{\left(k+\frac{n+1}{2}\right)\left(k-\frac{n-1}{2}\right)\right\}P_{m,j}^{k}(u)$$

i.e. $$\left\{K^2-\left(k+\frac{n+1}{2}\right)\left(k-\frac{n-1}{2}\right)\right\}P_{m,j}^{k}(u)=0,$$

since $P_{m,j}^{k}(u)$ is eigenvector, so
$$\left\{K^2-\left(k+\frac{n+1}{2}\right)\left(k-\frac{n-1}{2}\right)\right\}=0$$

(3.4) $$K=\left\{\left(k+\frac{n+1}{2}\right)\left(k-\frac{n-1}{2}\right)\right\}^{\frac{1}{2}}.$$

4. EXPLICIT FORM OF $P_{m,j}^{k}(u)$ EVALUATED FROM HARISH-CHANDRA IDENTITY

Harish-Chandra Identity (3.2) is a generating relation for Jacobi polynomial $P_{m,j}^{k}(u)$. In paper[1] Prof. Harish-Chandra has not given the explicit form of $P_{m,j}^{k}(u)$. In this article we are giving such form of $P_{m,j}^{k}(u)$ from Harish-Chandra Identity. Now from Identity (3.2) which can be written as

$$(4.1)\qquad \frac{\left(t_2\sin\frac{\theta}{2}+t_1\cos\frac{\theta}{2}\right)^{k-j}\left(t_2\cos\frac{\theta}{2}-t_1\sin\frac{\theta}{2}\right)^{k+j}}{\left\{(k-j)!(k+j)!\right\}^{\frac{1}{2}}} = \sum_{m=k}^{-k}\frac{t_1^{\,k-m}t_2^{\,k+m}}{\left\{(k-m)!(k+m)!\right\}^{\frac{1}{2}}}P_{m,j}^{k}(\cos\theta)$$

$$\text{L.H.S. of (4.1)} = \frac{\left(t_2\sin\frac{\theta}{2}+t_1\cos\frac{\theta}{2}\right)^{k-j}\left(t_2\cos\frac{\theta}{2}-t_1\sin\frac{\theta}{2}\right)^{k+j}}{\left\{(k-j)!(k+j)!\right\}^{\frac{1}{2}}}$$

$$=\frac{1}{\left\{(k-j)!(k+j)!\right\}^{\frac{1}{2}}}\left\{\sum_{r=0}^{k-j}\binom{k-j}{r}\left(t_2\sin\frac{\theta}{2}\right)^{k-j-r}\left(t_1\cos\frac{\theta}{2}\right)^{r}\times \sum_{s=0}^{k+j}\binom{k+j}{s}\left(t_2\cos\frac{\theta}{2}\right)^{k+j-s}\left(-t_1\sin\frac{\theta}{2}\right)^{s}\right\}$$

$$=\frac{1}{\left\{(k-j)!(k+j)!\right\}^{\frac{1}{2}}}\left\{\sum_{r=0}^{k-j}\sum_{s=0}^{k+j}\binom{k-j}{r}\binom{k+j}{s}t_1^{r+s}\,t_2^{2k-(r+s)}\times \left(\cos\frac{\theta}{2}\right)^{r+k+j-s}\left(\sin\frac{\theta}{2}\right)^{k-j-r+s}(-1)^{s}\right\},$$

put $r+s=k-m$ and $\cos\frac{\theta}{2}=\left(\frac{1+\cos\theta}{2}\right)^{\frac{1}{2}}$, $\sin\theta=\left(\frac{1-\cos\theta}{2}\right)^{\frac{1}{2}}$.

$$\text{L.H.S. of (4.1)} = \frac{1}{\left\{(k-j)!(k+j)!\right\}^{\frac{1}{2}}}\left\{2^{-k}\sum_{m=k}^{-k}\sum_{r=0}^{k-j}\binom{k-j}{r}\binom{k+j}{k-m-r}(-1)^{k-m-r}\,t_1^{k-m}\,t_2^{k+m}\times (1+\cos\theta)^{r+\frac{(j+m)}{2}}(1-\cos\theta)^{k-r-\frac{(j+m)}{2}}\right\}$$

$$=\sum_{m=k}^{-k}\frac{t_1^{\,k-m}t_2^{\,k+m}}{\left\{(k-m)!(k+m)!\right\}^{\frac{1}{2}}}2^{-k}\left\{\sum_{r=0}^{k-j}\frac{\left\{(k-m)!(k+m)!\right\}^{\frac{1}{2}}}{\left\{(k-j)!(k+j)!\right\}^{\frac{1}{2}}}\binom{k-j}{r}\binom{k+j}{k-m-r}\times\right.$$

$$(-1)^{k-m-r}(1+\cos\theta)^{r+\frac{(j+m)}{2}}(1-\cos\theta)^{k-r-\frac{(j+m)}{2}}\Bigg\}.$$

Compare L.H.S. and R.H.S. of equation (4.1) we get

$$P_{m,j}^{k}(\cos\theta)=2^{-k}\sum_{r=0}^{k-j}\frac{\{(k-m)!(k+m)!\}^{\frac{1}{2}}}{\{(k-j)!(k+j)!\}^{\frac{1}{2}}}\binom{k-j}{r}\binom{k+j}{k-m-r}(-1)^{k-m-r}\times$$
$$(1+\cos\theta)^{r+\frac{(j+m)}{2}}(1-\cos\theta)^{k-r-\frac{(j+m)}{2}}.$$

Hence we get

(4.2) $$P_{m,j}^{k}(u)=2^{-k}\sum_{r=0}^{k-j}\frac{\{(k-m)!(k+m)!\}^{\frac{1}{2}}}{\{(k-j)!(k+j)!\}^{\frac{1}{2}}}\binom{k-j}{r}\binom{k+j}{k-m-r}(-1)^{k-m-r}\times$$
$$(1+u)^{r+\frac{(j+m)}{2}}(1-u)^{k-r-\frac{(j+m)}{2}},$$

which is an explicit form of $P_{m,j}^{k}(u)$.

5. JACOBI POLYNOMIAL IN ZEMANIAN[3]

From Zemanian[3] chapter IX we know that the normalized eigenfunctions of the operator

(5.1) $$\eta=\left[w(x)\right]^{-1/2}D\left(x^{2}-1\right)w(x)D\left[w(x)\right]^{-1/2},$$

where $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$ and α, β are real numbers with $\alpha>-1$, $\beta>-1$ and $-1<x<1$ are given as

(5.2) $$\psi_{n}(x)=\left[\frac{w(x)}{h_{n}}\right]^{1/2}P_{n}^{(\alpha,\beta)}(x),\qquad n=1,2,3,......$$

where $$h_{n}=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)},$$

and the $P_{n}^{(\alpha,\beta)}(x)$ are the Jacobi polynomials defined as

(5.3) $$P_{n}^{(\alpha,\beta)}(x)=2^{-n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^{m},$$

these eigenfunctions $\psi_{n}(x)$ correspond to the eigenvalues $\lambda_{n}=n(n+\alpha+\beta+1)$.
Thus we have

$$\eta\psi_{n}(x)=\lambda_{n}\psi_{n}(x)$$

i.e. $$\left\{\left[w(x)\right]^{-1/2} D\left(x^2-1\right)w(x)D\left[w(x)\right]^{-1/2}\right\}\psi_n(x)=\left\{n(n+\alpha+\beta+1)\right\}\psi_n(x)$$

i.e. $$(1-x)^{-\alpha/2}(1+x)^{-\beta/2} D\left\{\left(x^2-1\right)(1-x)^{\alpha}(1+x)^{\beta} D\left\{(1-x)^{-\alpha/2}(1+x)^{-\beta/2}\psi_n(x)\right\}\right\}=\left\{n(n+\alpha+\beta+1)\right\}\psi_n(x)$$

i.e. $$-(1-x)^{-\alpha/2}(1+x)^{-\beta/2} D\left\{\frac{1}{2}(\alpha-\beta+\alpha x+\beta x)(1-x)^{\alpha/2}(1+x)^{\beta/2}\psi_n(x) (1-x)^{\frac{\alpha}{2}+1}(1+x)^{\frac{\beta}{2}+1}\psi_n'(x)\right\}=\left\{n(n+\alpha+\beta+1)\right\}\psi_n(x)$$

i.e. $$-\left\{\left(1-x^2\right)\psi_n''(x)-2x\psi_n'(x)+\left(\frac{\beta+\alpha}{2}\right)\psi_n(x) -\frac{\left\{(\beta-\alpha)-(\beta+\alpha)x\right\}^2}{4\left(1-x^2\right)}\psi_n(x)\right\}=\left\{n(n+\alpha+\beta+1)\right\}\psi_n(x)$$

(5.4) $$\left\{\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{\left\{\left(\frac{\beta-\alpha}{2}\right)-\left(\frac{\beta+\alpha}{2}\right)x\right\}^2}{\left(1-x^2\right)}+\left(\frac{\beta+\alpha}{2}\right)\right\}\psi_n(x) =-\left\{n(n+\alpha+\beta+1)\right\}\psi_n(x).$$

So $\psi_n(x)$ are the eigenvectors and $-\left\{n(n+\alpha+\beta+1)\right\}$ are eigenvalues of the differential operator

$$\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{\left\{\left(\frac{\beta-\alpha}{2}\right)-\left(\frac{\beta+\alpha}{2}\right)x\right\}^2}{\left(1-x^2\right)}+\left(\frac{\beta+\alpha}{2}\right).$$

6. COMPARISON OF TWO JACOBI POLYNOMIALS USED IN PAPER[1] AND ZEMANIAN[3]

According to Zemanian[3] from equation (5.4) put

(6.1) $$\left(\frac{\beta-\alpha}{2}\right)=m,\left(\frac{\beta+\alpha}{2}\right)=j,$$ we get

$$\left\{\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{\left\{m-jx\right\}^2}{\left(1-x^2\right)}+j\right\}\psi_n(x)=-n(n+2j+1)\psi_n(x)$$

$$\left\{\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{\{m-jx\}^2}{\left(1-x^2\right)}-j^2\right\}\psi_n(x)$$

$$=-n(n+2j+1)\psi_n(x)-\left(j+j^2\right)\psi_n(x)$$

(6.2) $$\left\{\left(1-x^2\right)\frac{d^2}{dx^2}-2x\frac{d}{dx}-\frac{\{m-jx\}^2}{\left(1-x^2\right)}-j^2\right\}\psi_n(x)=-(n+j)(n+j+1)\psi_n(x).$$

Comparing equation (3.3) and (6.2), with using (4.2), (5.2) and (6.1) we get

(6.3) $$\begin{cases} P_{m,j}^k(u)=(-1)^{j-m}\left(\frac{2}{2k+1}\right)^{\frac{1}{2}}\psi_n(x), u=x=\cos\theta, \\ k=n+j, m=\frac{\beta-\alpha}{2}, j=\frac{\beta+\alpha}{2} \end{cases}$$

(6.4) $$\begin{cases} \psi_n(x)=(-1)^{j-m}\left(\frac{2k+1}{2}\right)^{\frac{1}{2}}P_{m,j}^k(u), x=u=\cos\theta, \\ \alpha=j-m, \beta=j+m, n=k-j \end{cases}.$$

Where $\psi_n(x)$ be the normalized Jacobi polynomial[3].

7. CONCLUSION

We have come to the conclusion that the Jacobi polynomial[1] $P_{m,j}^k(u)$ is infact the value of $(-1)^{j-m}\left(\frac{2}{2k+1}\right)^{\frac{1}{2}}\psi_n(u)$ for $n=k-j$, $\alpha=j-m$ and $\beta=j+m$. This allows us to take k and j either both integers or both half of odd integers.

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