

A Mathematical Model for Flow and Diffusion through Stenotic Capillary-Tissue Exchange System

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Abstract. Fluid flow through modelled normal and stenosed capillary-tissue exchange system has highlighted issues that may have major applications for the study of diffusion phenomenon. The model incorporates modified Casson's fluid representation for the blood flow through an axially symmetric stenosis. Assessment of the severity of the disease could be made possible through the variation of a parameter named as retention parameter. Symmetry of the distribution of the wall shearing stress and resistive impedance and their growth with the developing stenosis is another important feature of this analysis. The concentration profiles and associated physiological diffusion variables involved in the analysis for normal and diseased state have been determined.

Keywords- Stenosis, Casson's fluid, diffusion, Resistive impedance, Wall shear stress, Retention parameter.

Introduction

Heart problem is one of the most common causes of death. Angina pectoris and myocardial infarction are two examples of heart diseases. Angina pectoris is the term used to describe the pain cause when the vessel is not carrying enough blood to the heart muscle. The pain occurs especially when the heart muscle needs more blood. In the case of myocardial infarction or heart attack, part of the heart muscle is destroyed. This causes severe pain in the chest that can lead to death. A healthy person is not aware of having a heart, or the important work it does in making the body works properly. The heart is the strongest organ in the body, and it works like a pump. The heart is like an engine, which can wear out, and breaks down completely, this is called cardiac arrest. Actually how likely someone is to have a heart attack depends on a number of things. The main cause that leads to a heart attack is atherosclerosis (see Figure 1).

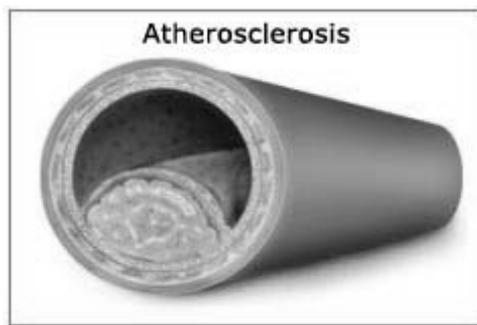


Figure 1. Atherosclerosis

Several authors [1-4] proposed various representative models for blood in narrow arteries. Chakravarty [5] mentioned that atherosclerosis occurs when the nature of blood flow changes from its usual state to a disturbed flow condition due to the presence of a stenosis in an artery. Stenosis is defined as a partial occlusion of the vessels caused by abnormal growth of tissues or the deposition of cholesterol as substances on the arterial wall.

A number of researchers have studied the flow of non-Newtonian fluids with various perspectives. Ronald L. Fournier [6] explained about the field of rheology concerns the deformation and flow behaviour of fluids, the prefix *rheo* is from Greek and refers to something that flows because of the particulate nature of blood. He expected the rheological behaviour of blood to be some what more complex than a simple fluid such as water. He mentioned that in order to understand the flow behaviour of blood, one must first define the relationship between shear stress and the shear rate. Ishikawa et al. [7] found that the non-Newtonian pulsatile flow through a stenosed tube is different from Newtonian flow. The non-Newtonian property strengthens the peaks of wall shear stress and wall pressure, weakens the strength of the vortex and reduces the vortex size and separated region. Therefore, he concluded that non-Newtonian flow is more stable than Newtonian flow. Taking into account the non-Newtonian behaviour of blood described by the Casson fluid model, Dash et.al. [8] studied the changed flow patterns in a narrow artery. Blood behaves like a non-Newtonian fluid at a low shear rate in smaller arteries under certain flow conditions [9,10,11,12,13]. The papers [14,15,16,17,18] provide a small sample of the research on non-Newtonian effects on blood flow. Perkkio and Keskinen [19] studied the effects of the concentration on viscosity and the effects of the concentration on blood flow through a vessel with stenosis and found it an important aspect from physiological point of view. Kang and Eringen [20] have also discussed the effects of the variation of the concentration of the suspended cells of the blood. Viscosity depending on the local variation of the concentration of the suspended cells has been introduced by Tandon et.al. [21,22].

Flow and diffusion through capillary-tissue exchange systems has been identified as one of the thrust areas of research in the recent past. In narrow capillaries, at times, the radial transport becomes much larger as compared to the axial transport and it contributes to the development of atherosclerotic plaques, greatly reducing the capillary diameter. The problem of flow and diffusion becomes much more difficult through a capillary with stenosis at some region. This research considers the study of blood flow and diffusion phenomenon through capillary-tissue exchange system in the presence of stenosis because it can cause the development of cardiovascular diseases. The

model incorporates modified Casson's fluid representation for the blood flow through a symmetric stenosis. Assessment of the severity of the disease could be made possible through the variation of a parameter named as retention parameter. Symmetry of the distribution of the wall shearing stress and resistive impedance and their growth with the developing stenosis is another important feature of this analysis. The concentration profiles and associated physiological diffusion variables involved in the analysis for normal and diseased state have been determined. Casson [2], Reiner et.al.[23], Merrill et.al. [24] have suggested that blood inhibits yield stress and behaves as a Casson model fluid at a shear rate equal to 0.1/sec. In fact Casson's relation is an empirical scalar equation and as such not applicable to general flow fields. Perktold et.al. [25] represented a generalized tensor form of Casson equation which satisfies the principle of material frame-indifference of continuum mechanics. This may be quite useful for blood flow situations in which the velocity field is not one-dimensional; however, for a circular tube the equation is quite adequate.

The Casson's relation is commonly written as

$$\tau^{1/2} = \tau_0^{1/2} + \mu^{1/2} \left(-\frac{du}{dr} \right)^{1/2}, \quad \tau \geq \tau_0 \quad (1)$$

$$\frac{du}{dr} = 0, \quad \tau \leq \tau_0 \quad (2)$$

where τ_0 is yield stress and μ denotes the viscosity coefficient.

Consider a laminar and steady flow of blood whose viscosity varies along the radial direction. In the capillary segment the geometry of the stenosis is given by (Figure. 2)

$$\frac{R(z)}{R_0} = 1 - A \left[L_0^{m-1} (z-d) - (z-d)^m \right], \quad d \leq z \leq d + L_0$$

$$= 1, \quad \textit{otherwise} \quad (3)$$

Here the parameter A is expressed as

$$A = \frac{\delta}{R_0 L_0^m} m^{m/(m-1)} \quad (4)$$

where δ denotes the maximum height of the stenosis at $z = d + \frac{L_0}{m^{m/(m-1)}}$ such that

$\frac{\delta}{R_0} \leq 1$. $R(z)$ and R_0 are the radius of capillary with and without stenosis respectively.

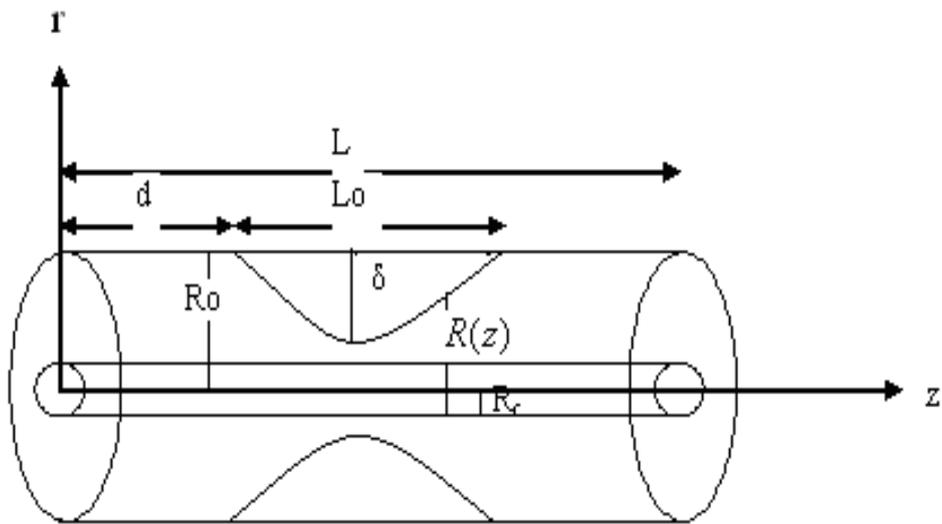


Figure 1. Flow Geometry of Stenosed Capillary

FORMULATION OF THE PROBLEM

Momentum equations are

$$0 = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial(r\tau)}{\partial r} \quad (5)$$

$$0 = -\frac{\partial p}{\partial r} \quad (6)$$

The concentration equation for the solute is expressed by

$$u \frac{\partial C}{\partial z} = D \left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} \right) \quad (7)$$

where C represents the concentration of the solute, u is the axial velocity and D the diffusion coefficient for the solute under consideration in the blood.

To solve the above system of equations, the following boundary conditions are introduced:

$$\left. \begin{aligned} \frac{\partial u}{\partial r} &= 0 && \text{at } r = 0 \\ u &= 0 && \text{at } r = R \\ p &= p_0 && \text{at } z = 0 \\ p &= p_L && \text{at } z = L \\ \frac{\partial C}{\partial r} &= 0 && \text{at } r = 0 \\ D \frac{\partial C}{\partial r} &= VNC && \text{at } r = R \end{aligned} \right\} \quad (8)$$

where N is the retention parameter.

SOLUTION OF THE PROBLEM

Solving for u from equations (1) and (2), using the boundary conditions given in (8), we obtain

$$\begin{aligned} u_1 &= -\frac{1}{4\mu} \frac{dp}{dz} \left[R^2 - r^2 - \frac{8}{3} R_c^{1/2} (R^{3/2} - r^{3/2}) + 2R_c (R - r) \right], \quad R_c \leq r \leq R \\ u_2 &= -\frac{1}{4\mu} \frac{dp}{dz} \left[R^2 - R_c^2 - \frac{8}{3} R_c^{1/2} (R^{3/2} - R_c^{3/2}) + 2R_c (R - R_c) \right], \quad 0 \leq r \leq R_c \end{aligned} \quad (9)$$

Here R_c is the radius of plug-flow region, defined by

$$\tau_0 = -\frac{dp}{dz} \frac{R_c}{2} \quad (10)$$

The volumetric rate of flow is defined as

$$Q = \pi \int_0^R r^2 \left(-\frac{du}{dr} \right) dr \quad (11)$$

By performing the integration of equation (11), using equations (9) and (10), one obtains

$$Q = \pi \frac{R^4}{8\mu} \left(-\frac{dp}{dz} \right) \left[1 - \frac{16}{7} \left(\frac{R_c}{R} \right)^{1/2} + \frac{4}{3} \left(\frac{R_c}{R} \right) - \frac{1}{21} \left(\frac{R_c}{R} \right)^4 \right] \quad (12)$$

Equation (12) can be rewritten as

$$Q = \pi \frac{R^4}{8\mu} \left(-\frac{dp}{dz} \right) f(\bar{y}) \quad (13)$$

where

$$f(\bar{y}) = 1 - \frac{16}{7} (\bar{y})^{1/2} + \frac{4}{3} (\bar{y}) - \frac{1}{21} (\bar{y})^4$$

and

$$\bar{y} = \frac{R_c}{R} < < 1$$

From equation (13), the pressure gradient is given by

$$-\frac{dp}{dz} = \frac{8\mu Q}{\pi R^4 f(\bar{y})} \quad (14)$$

The resistance to flow or the resistive impedance is denoted by λ and is defined as

$$\lambda = \frac{p_L - p_0}{Q} = \int_0^d \frac{dz}{\left(\frac{R(z)}{R_0}\right)^4 f(\bar{y}(z))} + \int_d^{d+L_0} \frac{dz}{\left(\frac{R(z)}{R_0}\right)^4 f(\bar{y}(z))} + \int_{d+L_0}^L \frac{dz}{\left(\frac{R(z)}{R_0}\right)^4 f(\bar{y}(z))} \quad (15)$$

or it can be written as

$$\lambda = \frac{L - L_0}{f_0} + \int_d^{d+L_0} \frac{dz}{\left(\frac{R(z)}{R_0}\right)^4 f(\bar{y}(z))} \quad (16)$$

Following Tandon and Pal [21] the apparent viscosity is expressed as

$$\frac{\mu_0}{\mu} = \frac{1}{\left(\frac{R(z)}{R_0}\right)^4 f(\bar{y}(z))} \quad (17)$$

The shearing stress at the wall can be defined as

$$\tau_s = \left\{ \tau_0^{1/2} + \left(-\mu \frac{du}{dr} \right)_{r=R(z)}^{1/2} \right\}^2 \quad (18)$$

To solve equation (7), we use the following non-dimensional quantities:

$$x = \frac{z}{L}, \quad \eta = \frac{r}{R_0}, \quad C_1 = \frac{C}{C_0}, \quad \eta_c = \frac{R_c}{R_0}, \quad R' = \frac{R}{R_0} \quad (19)$$

so that the equation (7) takes the form

$$\frac{\partial^2 C_1}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial C_1}{\partial \eta} = \frac{\nu R_0^2}{D_1 L} \frac{\partial C_1}{\partial x} \quad (20)$$

together with the boundary conditions

$$\begin{aligned} \frac{\partial C_1}{\partial \eta} &= 0, \quad \text{at } \eta = 0 \\ D_1 \frac{\partial C_1}{\partial \eta} &= VNC_1, \quad \text{at } \eta = \frac{R}{R_0} \end{aligned} \quad (21)$$

where $\nu = u - \bar{u}$ and N is the retention parameter and

$$\bar{u} = \frac{R_0^2}{4\mu L} \left(-\frac{dp}{dx} \right) \left[R'^2 + 2R'\eta_c - \frac{8}{3}\eta_c^{1/2}R'^{3/2} - \frac{1}{3}\eta_c^2 \right] \quad (22)$$

The solution for C_1 from (20), subject to the boundary conditions (21), is given by

$$C_1 = \frac{R_0}{4\mu L^2 D_1} \left(-\frac{dp}{dx} \right) \frac{\partial C_1}{\partial x} \left[\frac{R'^2 \eta^2}{4} - \frac{\eta^4}{16} \frac{8}{3} \eta_c^{1/2} \left(\frac{R'^{3/2} \eta^2}{4} - \frac{4\eta^{7/2}}{49} \right) + 2\eta_c \left(\frac{R' \eta^2}{4} - \frac{\eta^3}{49} \right) \right] - \frac{\bar{u} \eta^2 R_0^2}{4D_1 L} \frac{\partial C_1}{\partial z} + A \quad (23)$$

$$\text{where } A = \left[\frac{R_0^4}{4\mu L^2} \left(-\frac{dp}{dz} \right) \left\{ \frac{R'^3}{4} \left(1 - \frac{VNR_0 R'}{D_1} \right) - \frac{2}{7} \eta_c^{1/2} R'^{5/2} \left(2 - \frac{11 VNR_0 R'}{D_1} \right) + \frac{\eta_c R'^2}{3} \left(1 - \frac{5VNR' R_0}{6D_1} \right) \right\} - \frac{R_0^2}{L} \bar{u} R' \left(1 - \frac{VNR_0 R'}{2D_1} \right) \right] \frac{\partial C_1}{\partial z}.$$

RESULTS AND DISCUSSIONS

In this paper the concentration profiles and associated physiological diffusion variables involved in the analysis for normal and diseased system associated with stenosis due to the local deposition of lipids have been determined. Such models may help in identification, diagnosis and treatment of many cardiovascular disorders. The results are shown in the figures 3 to 10 by taking the value of parameters based on experimental data in a capillary. The values of parameters used are:

$$L = 5\text{cm} ; m = 2, 3, 4, \dots, 9$$

$$\frac{L}{L_0} = 0.02, 0.04, 0.06, \dots, 0.14$$

$$\frac{\delta}{R_0} = 0.2, 0.3, 0.4, \dots, 0.8$$

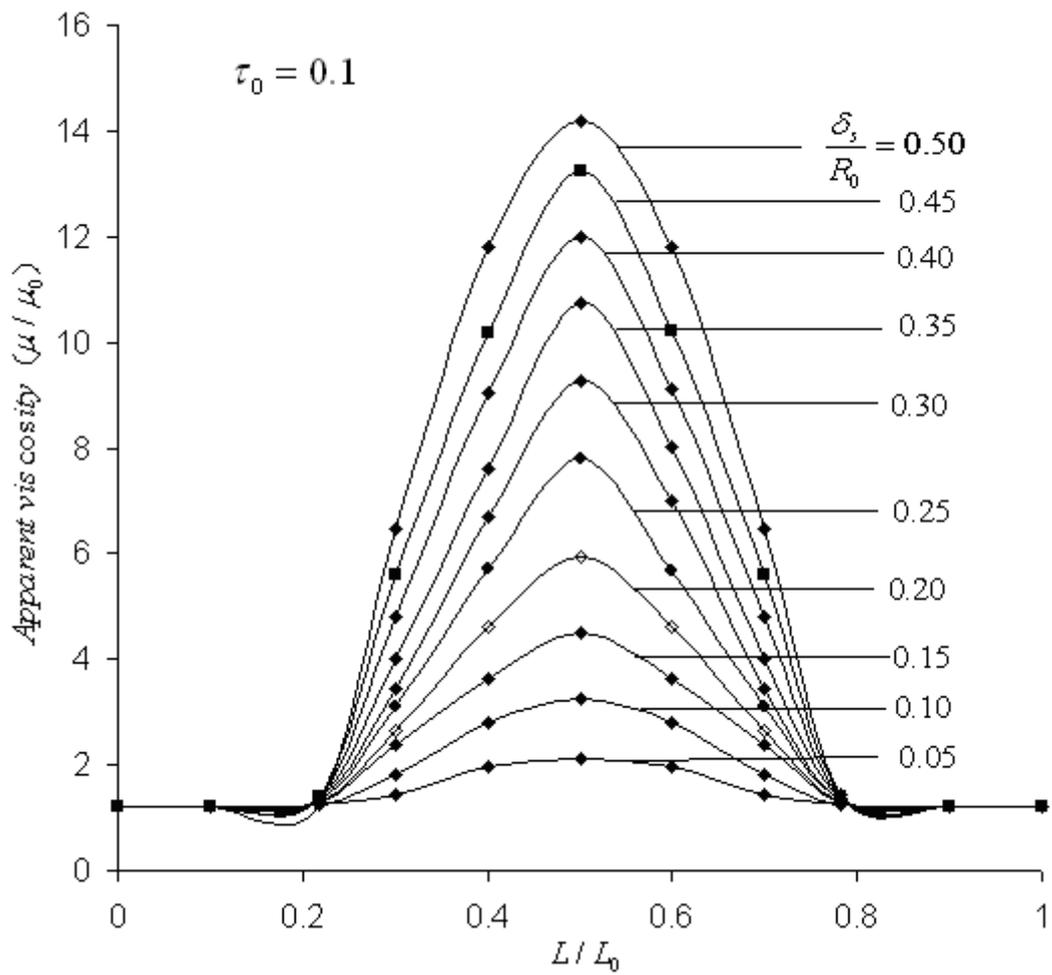


Figure 3. Variation of Apparent Viscosity (μ / μ_0) with (L / L_0) for different values of (δ / R_0)

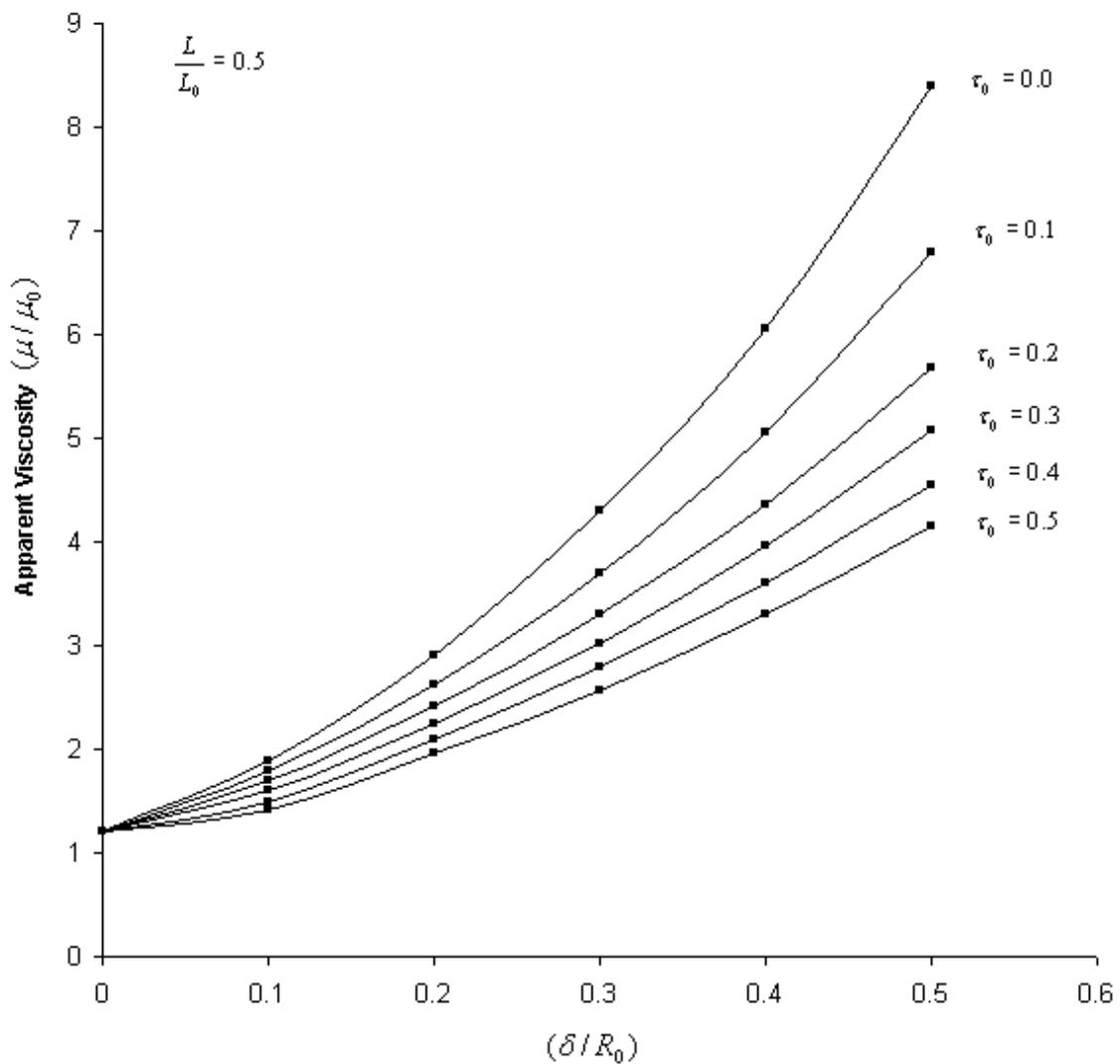


Figure 4. Variation of Apparent viscosity(μ / μ_0) with (δ / R_0) for different values of τ_0

It may be observed from figure 3 and 4 that the apparent viscosity increases as the stenosis grows. But in the absence of stenosis above is not true. In a normal artery the apparent viscosity decreases as the radius decreases, which is referred to as Fahraeus-Lindquist effect.

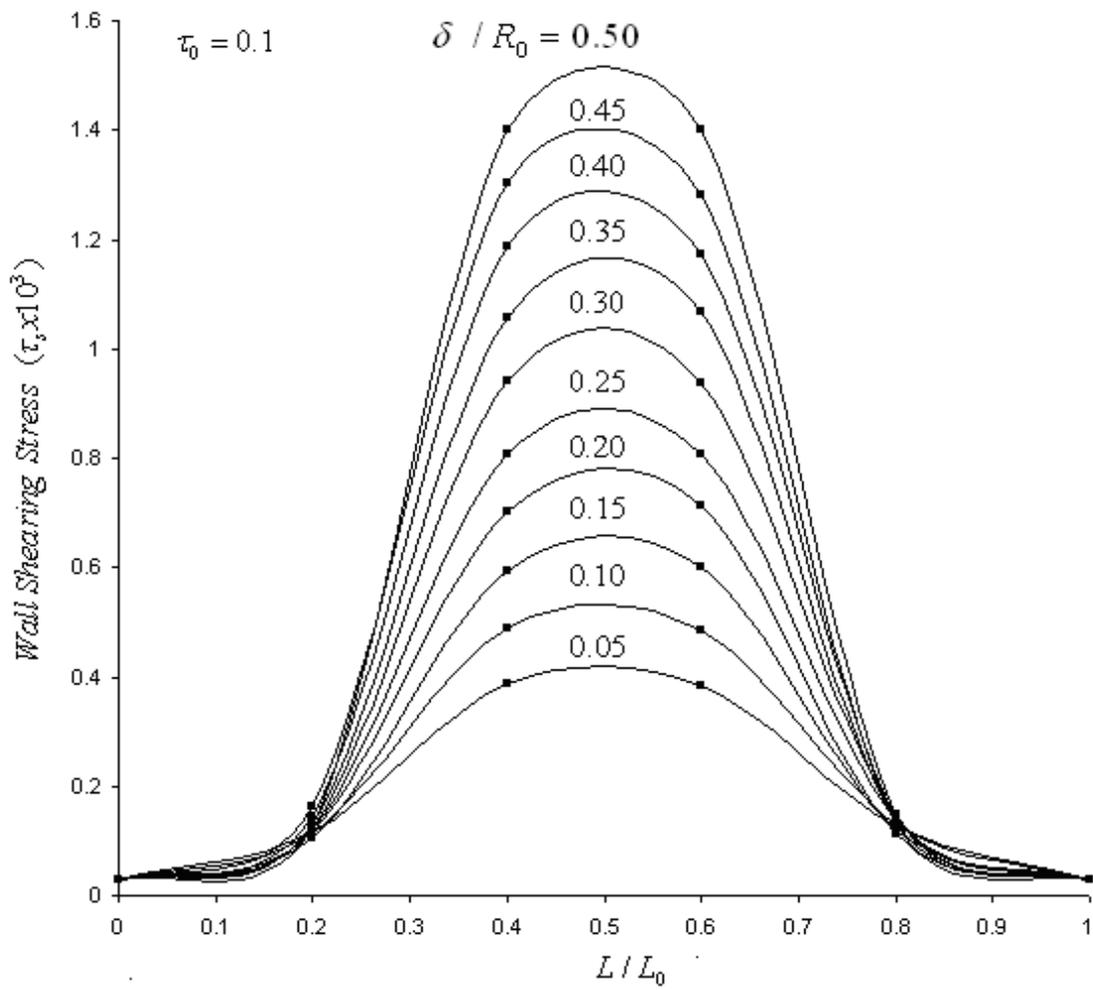


Figure 5. Variation of Wall shearing stress (τ_s) with (L/L_0) for different values of (δ/R_0)

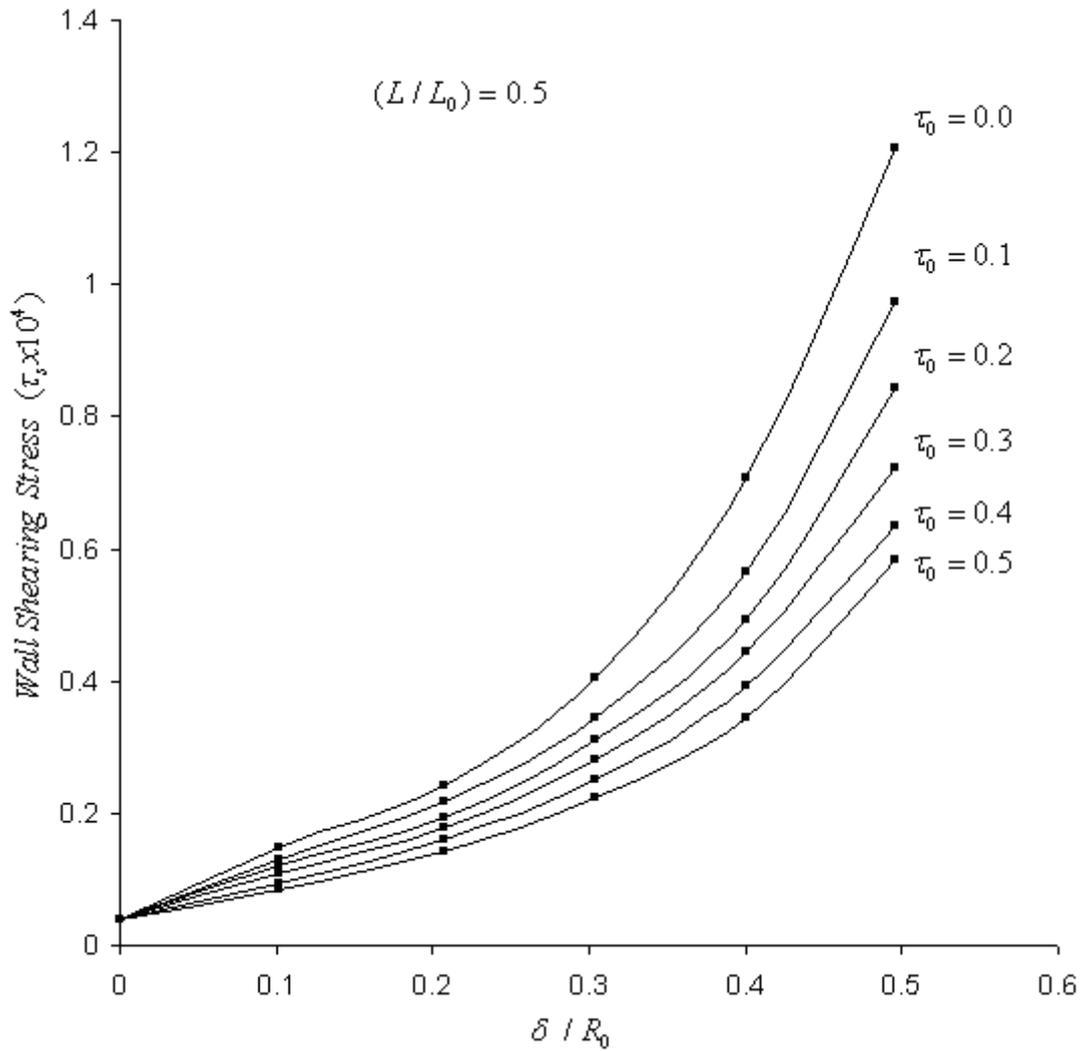


Figure 6. Variation of Wall shearing stress (τ_s) with (δ / R_0) for different values of (τ_0)

Restricted to laminar flow conditions only, one may observe, from fig. 5 and 6, that the shearing stress is symmetrically distributed in the stenotic region and is maximum at the minimum depth of the stenosis. The wall shear shearing stress is uniformly distributed in non-stenotic regions which is an obvious well known fact.

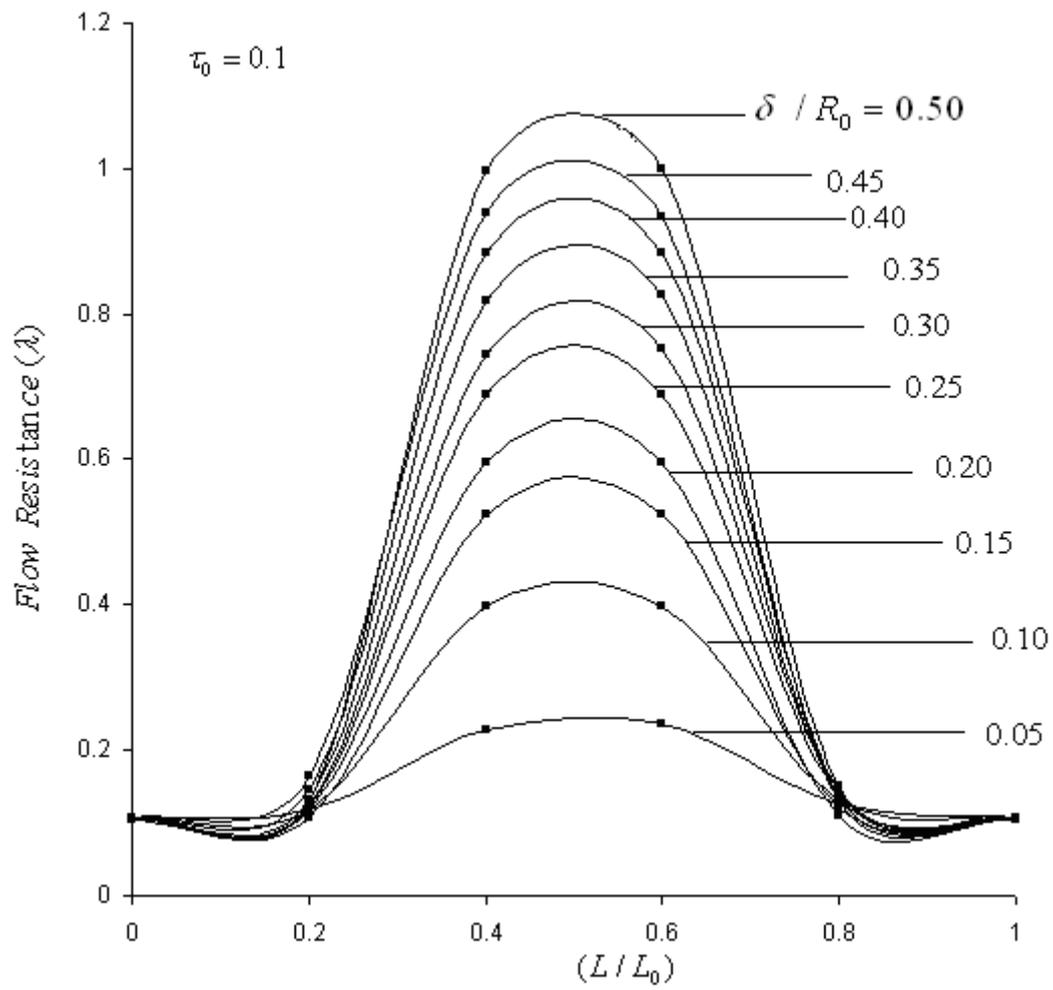


Figure 7. Variation of Resistance to flow (λ) with (L/L_0) for different values of (δ/R_0)

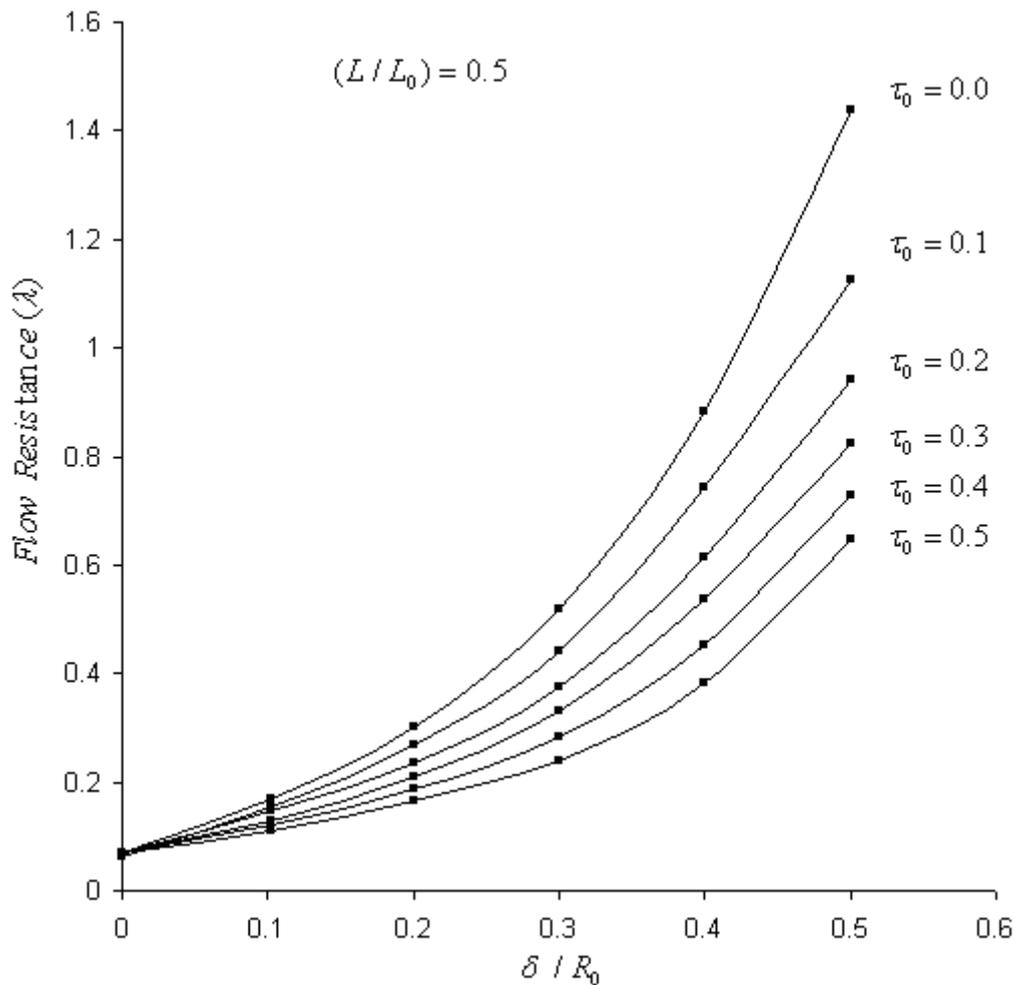


Figure 8. Variation of Resistance to flow (λ) with (δ / R_0) for different values of (τ_0)

Figure 7 and 8, explain that the flow resistance is symmetrically distributed in the stenotic region and uniform in non-stenotic regions. Thus the results for resistance to flow and wall shearing stress are similar in stenotic and non-stenotic regions. The symmetry of the curves may be attributed to the assumed geometrical configuration in addition to the maintenance to laminar flow regime.

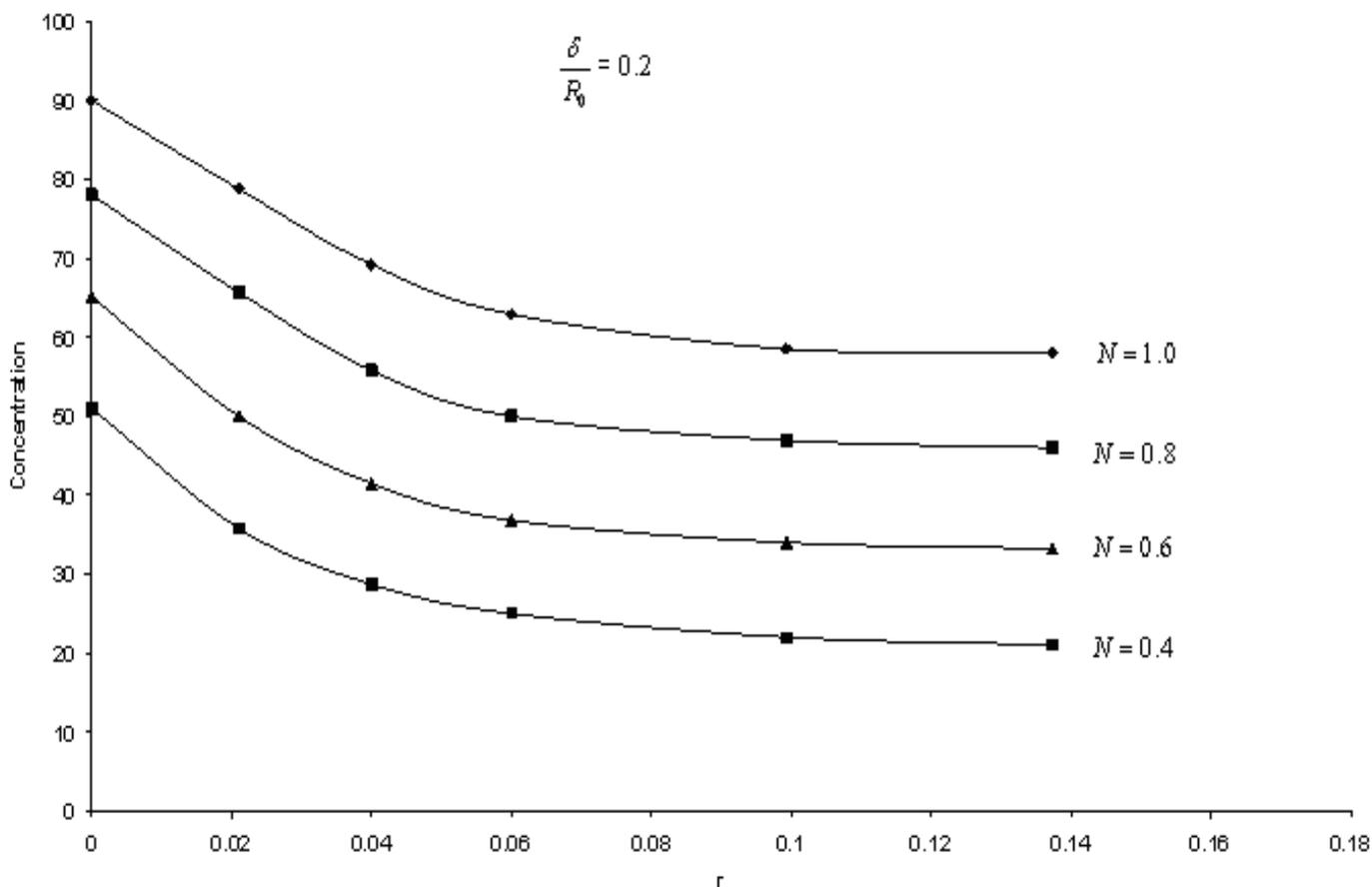


Figure 9. Concentration profiles for different values of Retention parameter N

Figure 9. reveals the effect of retention parameter (N) on concentration in capillary. Increasing values of N describe the increase in retention of the solutes within the capillary. $N=1$ implies the complete retention, i.e., no solute or fluid diffuses and as the retention parameter decreases from 1 to 0.4 more solute diffuses, which in turn, decreases the solute concentration in the capillary region. The variation of the values of retention parameter in the stenotic region may also be associated with the type of plaques deposited on the walls: calcified, fibrous or fatty plaques.

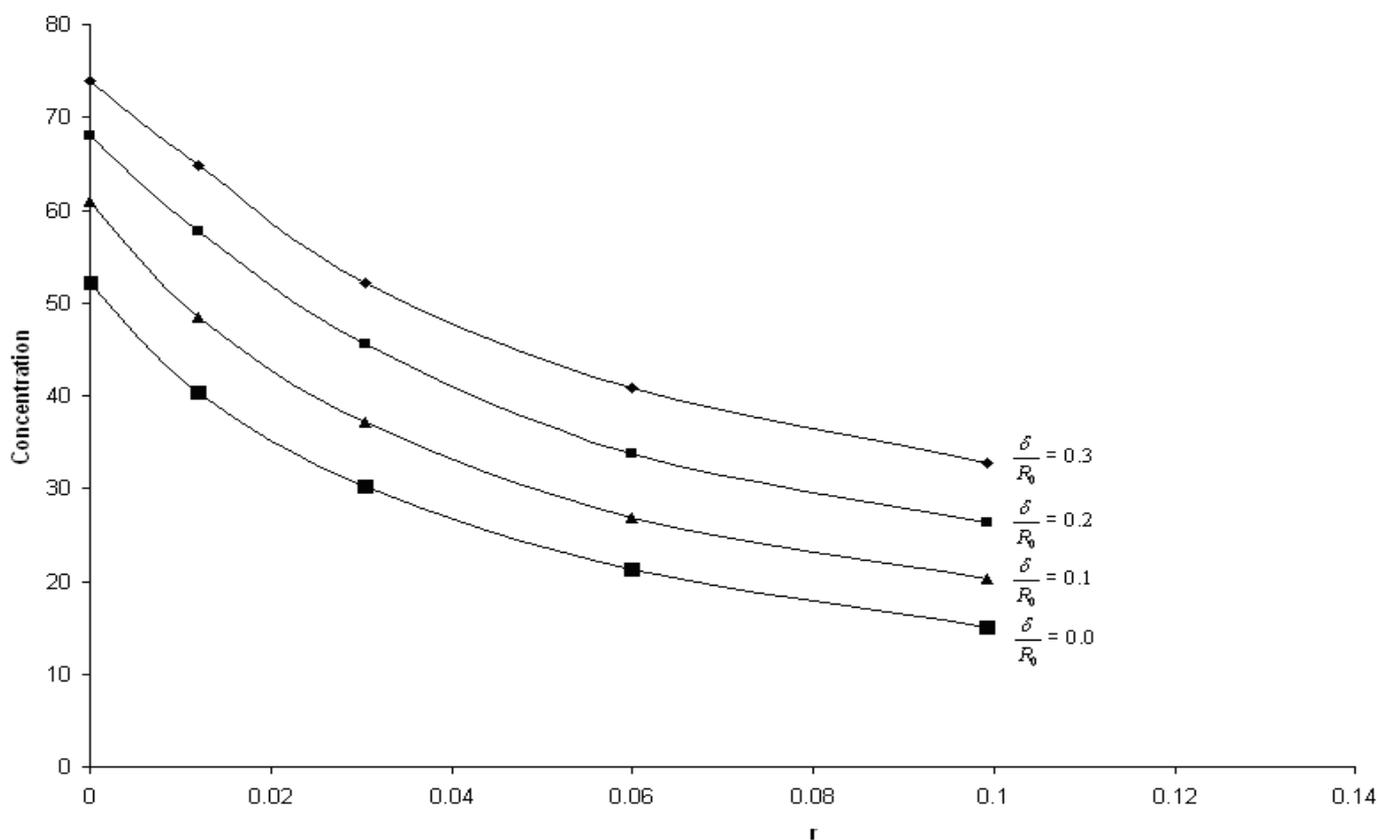


Figure 10. Concentration profiles for different values of Stenosis size (δ / R_0)

Figure 10. shows the diffusion of large and small molecular weight nutrients within the capillary for different values of stenosis size (δ / R_0). Large molecular weight nutrients face more resistance to diffuse into the tissue and therefore the cells of the deeper region are deprived of getting sufficient nutrition.

Concluding Remarks

This model has incorporated simultaneous dispersion of solute in normal and stenotic capillary depending on various parameters including retention parameters. The results are more encouraging and correlating well with the experimental observation that deeper region cells are deprived of the nutrients in the stenotic region. More experimental results are required for further development from clinical point of view.

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