APPROXIMATION OF ALMOST LIPα FUNCTION BY K^{λ}-SUMMABILITY METHOD

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Abstract: A new theorem on the approximation of almost Lipα function by K^{λ}– summability method of Fourier series.

1.INTRODUCTION

In 1935, first time, Karamata [5] introduced K^{λ} – summability method. In 1963, special case for λ=1 of this method has been reintroduced by Lotosky [8].Further studies of this and similar methods took place due to contribution of Agnew [1] on evalution of series. Vuĉkoviĉ [14] studie Fourier series by Karamata method (K^{λ}). Karthal [4] extended Vuĉkoviĉ result. Working in the same direction Ojha [9], Tripathi & Lal [13], Lal [17] Lal & Pratap [6] have generalised Katahal's result on K^{λ} -summability of Fourier series under general conditions. The degree of approximation by Cesáro mean and Nörlund means of a function f∈ Lipα has been determined by number of researches like Alexits [2], Sahney & Goel [12], Chandra [3], Qureshi [10], Qureshi & Nema [11]. But till now nothing seems to have been done for the degree of approximation of a function belonging to almost Lipschitz class, denoted by $Li^{a}p\alpha$, by K^{λ} –summability means. Almost Lipα class is a generalization of Lipα class. The purpose of this paper is to determine the approximation of almost Lipschitz function by Karamata (k^{λ}) method.

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2. DEFINITION AND NOTATION

Let $0 < \alpha \leq 1$ and let $f : R \to R$ be almost Lipschitz of order α, $f \in \overset{a}{\text{Lip}}\, \alpha$, in the sense that there is a constant $M = M_f \geq 0$, and for each $x \in R$ there is a subset $A_x \subset [0, \pi/2]$ of measure zero, such that $t \in \left(0, \frac{\pi}{2}\right) / A_x$ implies.

$$|f(x+2t) - f(x)| \leq M\, t^{\alpha}$$

Now, we assume further that the $\overset{a}{\text{Lip}}\, \alpha$ function f is 2π – periodic on R and Lebesgue integrable on (-π,π). Then its Fourier series is given by

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

Where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \cos nu \, du$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \sin nu \, du \qquad (n=1,2,3\ldots\ldots.)$$

The degree of approximation of a function f: R →R by a trigonometric polynomial T_n of order n is defined by [Zygmund(15)],

$$\|t_n - f\|_{\infty} = \text{Sup} \{| t_n(x) - f(x) | : x \in R\} \tag{2.2}$$

Let us define, for n = 0,1,2,3 ……, the number $\begin{bmatrix} n \\ m \end{bmatrix}$, $0 \leq m \leq n$, by

$$x(x+1)(x+2)\ldots\ldots\ldots\ldots\ldots\ldots(x+n-1) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m,$$

i.e
$$\frac{\Gamma(x+N)}{\Gamma x} = \prod_{v=0}^{n-1} (x+v) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m \tag{2.3}$$

The numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ are known absolute values of Stirling numbers of the first kind.

Let $\{S_n\}$ be the sequence of partial sums of partial sums of an infinite series $\sum a_n$ and let us write.

$$S_n^{\lambda} = \frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m S_m \tag{2.4}$$

to denote the n^{th} K^{λ}- mean of order $\lambda > 0$, if $s_n^{\lambda} \to S$ as $n \to \infty$, where S is a fixed finite quantity, then the sequence $\{S_n\}$ of the series $\sum a_n$ is said to be summable by Karamata method K^{λ} order $\lambda > 0$, to the sum S and we can write.

$$s_n^{\lambda} \to s(K^{\lambda}) \quad \text{as } n \to \infty$$

We use following notations:

$$\Phi(t) = f(x+2t) + f(x-2t) - 2f(x), \quad (2.5)$$

$$K_n(t) = \frac{\Gamma\lambda}{\pi} \frac{\sum_{m=0}^{n} \left[{n \atop m}\right] \lambda^m \sin(2m+1)t}{\Gamma(\lambda+n)\sin t} \quad (2.6)$$

3. MAIN THEOREM

Quite good amount of works are known the degree of approximation of a function f∈ $\overset{a}{Lip}\,\alpha$ of Fourier series by Cesáro and Nörlund summability means. The purpose of this paper is to determine the approximation of almost Lipschitz function by K^{λ} –summability method in the following form.

Theorem: If f :R → R is 2π periodic, Lebesgue integrable on $(-\pi,\pi)$ and is almost $Li^{a}p\alpha$, $f \in \dot{Lip}$ then the approximation of a function f by K^{λ} –means

$S_n^{\lambda} = \frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \left[{n \atop m}\right] \lambda^m S_m$ of Fourier series (2.1) satisfies

$$\left\|s_n^{\lambda} - f\right\|_{\infty} = O\left[\frac{\log(n+1)e}{(n+1)^{\alpha+1}} + \frac{1}{(n+1)^{\lambda}}\right], \qquad \text{for } n=0, 1, 2, 3\ldots\ldots$$

4. LEMMA

For the proof of our theorem following lemma is required

Lemma. (Vuĉkoviĉ 1965) Let $\lambda>0$ and $0<t<\frac{\pi}{2}$.

Then

$$\left\{\frac{\operatorname{Im}\Gamma(\lambda e^{2it}+n)}{\Gamma(\lambda\cos 2t+n)\sin t}\right\} = \frac{|\sin(\lambda\log(n+1))\sin 2t|}{\sin t} + O(1),$$

as $n \to \infty$ uniformly in t.

5. PROOF OF THE THEOREM

The m^{th} parital sum of the Fourier series (2.1) at the point t =x is given by

$$S_m - f(x) = \frac{1}{\pi}\int_0^{\pi/2} \frac{\sin(2m+1)}{\sin t} \phi(t)dt \quad (5.1)$$

Then

$$\frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \left[{n \atop m}\right] \lambda^m (S_m - f(x) = \frac{1}{\pi}\int_0^{\pi/2} \frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \left[{n \atop m}\right] \lambda^m \frac{\sin(2m+1)t}{\sin t}) \phi(t)dt$$

or $$\frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \left[{n \atop m}\right] \lambda^m S_m - \frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \left[{n \atop m}\right] \lambda^m f(x) = \int_0^{\pi/2} K_n(t)\phi(t)dt$$

$$s_n^{\lambda} - f(x) = \int_0^{\frac{\pi}{2}} K_n(t)\ \phi(t)dt \quad \text{by} \quad (2.6)$$

$$= \left[\int_0^{\frac{\pi}{2}} K_n(t) || \phi(t)\ | dt\right]$$

$$= [\{\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi/2}\} |K_n(t)|\ |\phi(t)|\ dt]$$

$$= [\int_0^{1/(n+1)} | Kn(t) ||\ \phi(t) |\ dt] + [\int_{1/(n+1)}^{\pi/2} | Kn(t) ||\ \phi(t) |\ dt]$$

$$= I_1 + I_2 \tag{5.2}$$

Now,

$$K_n = \frac{\Gamma\lambda}{\pi} \frac{\sum_{m=0}^{n} \left[\begin{matrix} n \\ m \end{matrix}\right] \lambda^m \sin(2m+1))t}{\Gamma(\lambda+n)\sin t}$$

$$= \quad \text{Imaginary part of} \left\{\frac{\sum_{m=0}^{n} \left[\begin{matrix} n \\ m \end{matrix}\right] \lambda^m e^{i(2m+1)t}}{\Gamma(\lambda+n)\sin t}\right\}$$

$$= \quad I_p \left\{\frac{\sum_{m=0}^{n} \left[\begin{matrix} n \\ m \end{matrix}\right] \lambda^m e^{2imt+e^{it}})}{\Gamma(\lambda+n)\sin t}\right\}$$

$$= I_p \left\{\frac{\sum_{m=0}^{n} \left[\begin{matrix} n \\ m \end{matrix}\right] (\lambda e^{2it})^m e^{it}}{\Gamma(\lambda+n)\sin t}\right\}$$

$$= \quad I_p \left\{\frac{\frac{\Gamma(\lambda e^{2it}+n)}{\Gamma\lambda e^{2it}}}{\Gamma(\lambda+n)\sin t}\right\}$$

$$= \left\{\frac{(\cos t + i\sin t)\frac{\Gamma(\lambda e^{2it}+n)}{\Gamma\lambda e^{2it}}}{\Gamma(\lambda+n)\sin t}\right\}$$

$$= \frac{\cos t I_p \frac{\Gamma(\lambda e^{2it}+n)}{\Gamma\lambda e^{2it}} + \sin t\ Real\ part\ of \frac{\Gamma(\lambda e^{2it}+n)}{\Gamma\lambda e^{2it}}}{(\lambda+n)\sin t}$$

$$= O\left[\frac{I_p\{\Gamma(\lambda e^{2it}+n)\}}{\Gamma(\lambda+n)\sin t}\right] + O\left[\frac{\text{Real part of}\{\Gamma(\lambda e^{2it}+n)\}}{\Gamma(\lambda+n)}\right]$$

$$K_n(t) = O\left[\frac{\frac{\Gamma(\lambda\cos 2t+n)\ I_p\{\Gamma(\lambda e^{2it}+n)\}}{\Gamma(\lambda+n)}}{\Gamma(\lambda\cos 2t+n)}\right] + O\left[\frac{\Gamma(\lambda\cos 2t+n)}{\Gamma(\lambda+n)}\right]$$

For $0 < t < \frac{1}{(n+1)}$

$$\frac{\Gamma(\lambda\cos 2t+n)}{\Gamma(\lambda+n)} = O\left[n^{-\lambda^{(1-\cos 2t)}}\right]$$

$$= O\left[e^{-\frac{\lambda}{2}^{(2t)^2 log(n+1)}}\right]$$

$$= O\left[e^{-2\lambda t^2 \log(n+1)}\right]$$

Since for, $0< t<\frac{1}{n+1}$, $0 < 1 - 1\cos 2t < 2t^2$

therefore,

$$K_n(t) = O\left[\frac{e^{-2\lambda t^2\log(n+1)}\, I_p\,\{\Gamma(\lambda e^{2it}+n)\}}{\Gamma(\lambda\cos 2t+n)\ \sin t}\right] + O\left[e^{-2\lambda t^2\log(n+1)}\right], \text{ for } 0 < t < \frac{1}{n+1}$$

$$= O\left[e^{-2\lambda t^2\log(n+1)}\left\{\frac{|\{\sin t[\lambda log(n+1)\sin 2t]\}|}{\sin t}\right\} + O(1)\right] + O\left[e^{-2\lambda t^2\log(n+1)}\right]$$

by Lemma 4

$$= O\left[\frac{e^{-2\lambda t^2\log(n+1)}}{\sin t}\{|\{\sin t\,[\lambda log(n+1)\sin 2t]\}|\}\right] + O\left[e^{-2\lambda t^2\log(n+1)}\right] + O\left[e^{-2\lambda t^2\log(n+1)}\right]$$

$$= O\left|e^{-2\lambda t^2\log(n+1)}(2\lambda\log(n+1)\right| + O\left|e^{-2\lambda t^2\log(n+1)}\right|$$

$$= O\left[e^{-2\lambda t^2\log(n+1)}(\log(n+1)1\right]$$

$$K_{n(t)} = O(log(n+1)e) \qquad \left(as\ e^{-2\lambda t^2\log(n+1)}\right) \le 1 \tag{5.3}$$

and

$$K_{n(t)} = O\left\{\frac{1}{(n+1)^{\lambda}t}\right\},\quad for\quad \frac{1}{n+1} < t < \frac{\pi}{2} \tag{5.4}$$

Also ,$|f(x+2t) - f(x)| \le Mt^{\alpha}$ $since\ f \in Lip\ \alpha$

Then

$$|\phi(t)| = |f(x+2t) + f(x-2t) - 2f(x)|$$

$$\le |f(x+2t) - f(x) + f(x-2t) - f(x)|$$

$$= Mt^{\alpha} + Mt^{\alpha}$$

$$= 2Mt^{\alpha}$$

$$= O(t^{\alpha})$$

Thus $\phi \in \overset{a}{\text{Lip}}\, \alpha$

Now

$$I_1 = \int_0^{1/n+1} |K_n(t)|\,|\phi(t)|dt$$

$$= O(\lambda log(n+1)e) \int_0^{1/(n+1)} |\phi(t)|\,dt$$

$$= O(\lambda log(n+1)e) \int_0^{1/(n+1)} 2Mt^{\alpha}\,dt$$

$$= O(\lambda log(n+1)e) 2M \left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{(n+1)}}$$

$$= O\left[\frac{2M\lambda(\lambda log(n+1)e)}{(\alpha+1)(n+1)^{\alpha+1}}\right]$$

$$I_1 = O\left[\frac{\log (n+1)e}{(n+1)^{\alpha+1}}\right] \tag{5.6}$$

Next,

$$I_2 = \int_{\frac{1}{n+1}}^{\frac{\pi}{2}} |K_n(t)|\,|\phi(t)|dt$$

$$= \int_{\frac{1}{n+1}}^{\frac{\pi}{2}} \frac{1}{(n+1)^{\lambda} t} |\phi(t)|\,dt, \quad by\ (5.4)$$

$$= O\left(\frac{1}{(n+1)^{\lambda}}\right) \int_{\frac{1}{n+1}}^{\frac{\pi}{2}} \frac{|\phi(t)|}{t} dt$$

$$= O\left(\frac{1}{(n+1)^{\lambda}}\right) \int_{\frac{1}{n+1}}^{\frac{\pi}{2}} 2Mt^{\alpha-1}\,dt, \quad by\ (5.5)$$

$$= O\left(\frac{1}{(n+1)^{\lambda}}\right) 2M \left[\frac{t^{\alpha}}{\alpha}\right]_{\frac{1}{n+1}}^{\frac{\pi}{2}}$$

$$= O\left(\frac{1}{(n+1)^{\lambda}}\right) 2M \left[\frac{(\pi/2)^{\alpha}}{\alpha} - \frac{1}{\alpha(n+1)^{\alpha}}\right]$$

$$\leq O\left[\frac{2M}{\alpha}\left(\frac{\pi}{2}\right)^{\alpha} \frac{1}{(n+1)^{\lambda}}\right] + O\left[\frac{2M}{\alpha}\,\frac{1}{(n+1)^{\alpha+\lambda}}\right]$$

$$\leq O\left(\frac{1}{(n+1)^{\lambda}}\right) + O\left(\frac{1}{(n+1)^{\alpha+\lambda}}\right)$$

$$\leq O\left(\frac{1}{(n+1)^{\lambda}}\right) + O\left(\frac{1}{(n+1)^{\lambda}}\right)$$

$$I_2 = O\left(\frac{1}{(n+1)^{\lambda}}\right) \tag{5.7}$$

Collecting (5.2), (5.6) and (5.7), we have

$$\left|s_n^\lambda - f(x)\right| \;=\; O\left[\frac{\log(n+1)e}{(n+1)^{\alpha+1}} + \frac{1}{(n+1)^\lambda}\right]$$

$$= Sup\left\{\left|s_n^\lambda(x) - f(x)\right|\right\}$$

Hence

$$\left\|s_n^\lambda - f(x)\right\|_\infty = O\left[\frac{\log(n+1)e}{(n+1)^{\alpha+1}} + \frac{1}{(n+1)^\lambda}\right]$$

This completes the Proof of the theorem.

Remark: Result similar to the main theorem may be obtained for a functions f€Lipα.

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