Three layer model for blood flow in small vessels

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Abstract

A three layer model consisting of a core and a region of suspension of the erythrocytes in plasma (fluid) of viscosity $\mu_1 = \mu_0(1+\alpha c)$ and a peripheral layer of cell free plasma layer has been proposed in this paper to represent blood flow in small capillary and compared with the two fluid model (Casson fluid model) and particle fluid mixture model of V. P. Srivastava. For three layer as well as casson fluid model equation of motion and equation of continuity for different regions are solved using analytical methods. The analytical results obtained in the proposed model for effective viscosity, velocity profile and flow rate for different values of hematocrit has been presented and discussed. Results are similar to those obtained by Particle fluid mixture of V.P.srivastava.

Introduction

Blood is essential to maintain life. It transports oxygen and nutrients to all parts of the body, relays chemical signals and moves metabolic waste to the kidneys for elimination. Yet despite more than 150 years of close study, a concise, predictive model of blood flow is still lacking. A quantitative model of blood flow is important not only as it relates to clinical diagnosis of disease, but as an integral component of models of more complex structures like the brain. Further, proper design of artificial organs demands a thorough understanding of blood rheology in order to avoid flow stagnation and clot formation.

The study of blood flow through microcirculation is the subject of scientific research about a couple of centuries. Blood from mechanical point of view can be considered as a neutrally buoyant suspension of erythrocytes in a Newtonian liquid called plasma. The two phase nature of blood as a suspension becomes important as the diameter of tube decreases. When the diameter of the tube is less than 500 μm the dimension of RBC are no longer negligible as compared to the tube size. The blood flow in these tubes has to be treated as two phase fluid. Since blood is a suspension of red cells in plasma, it behaves as a non Newtonian fluid at low shear rate and the yield stress is non zero at that stage.

Several researchers [Casson 1959, Hayness1959, Charm and Kurland 1964, Eringen 1964, Gupta et.al. 1982, Chaturani and Upadhyay 1984] have worked for blood flow in small blood vessels But some investigator Haynes and Burton (1959), Merill et.al. (1963). Charm and Kurland (1965) Hurshey et.al. (1964). Cokelet (1972) and Lih (1975) noticed that blood being a suspension of corpuscles, behave like a non Newtonian fluid at low shear rates. Bugliarello and Sevilla (1970). Cokelet (1972) and Thurston 1989 have shown experimentally that for blood flowing through small vessels there is a cell free plasma layer and a core region of suspension of all the erythrocytes. Haynes (1960) presented a two fluid theoretical model for blood flow consisting of a core region consisting of all the erythrocytes as a homogeneous Newtonian viscous fluid and a cell free plasma layer as a Newtonian fluid of constant viscosity. Viscosity is not a relevant property of blood when blood flow in capillaries is being studied. Blood does not flow as a homogeneous suspension in capillaries as it does in larger vessels; rather, it flows as a two phase system cells and plasma. Bugliarello and Sevilla (1970) presented blood in small diameter tubes by a two layered model assuming peripheral and core fluids as Newtonian fluids of different viscosities. Chaturani and Upadhyay considered the blood flow in small diameter tubes including the two layered model of micropolar and couple stress fluids. Wang and Bassingthwainghte (2003) presented a two layered model of Haynes (1960) and Sharan and Popel (2001) to discuss the flow of blood in narrow curvewd tubes. Gupta et al has been developed a three layer mathematical model and tested for calculating the velocity profile and wall layer thickness for the flow of blood and other particulate suspension in narrow tubes. Chaturani and Biswas have been considered three layered model includes Coutte flow of blood.

A few mathematical models which regard the blood in microvessels as a two component fluids have already been proposed. We present here a new mathematical model of the blood flow in microvessels. The model consists of plug region which is surrounded by suspension of red cells in plasma and cell free peripheral plasma layer. The viscosity of the suspension is considered to be depend on concentration of red cells i.e. $\mu_1{}' = \mu_0(1+\alpha' c')$ where μ_0 is the viscosity of plasma, α is the concentration coefficient. The result of the three layer model has been considerable with the two layer fluid model and particle fluid mixture model of V.P. Srivastava.

Formulation of the Problem

As described above we are presenting here two different models three layer model and casson fluid model.

Three layer model

The basic functional unit considered here includes a cylindrical capillary of radius R_2 and length ℓ. Blood is represented by a three layer model consisting of a core layer of all erythrocytes, central layer of suspension of red cell in plasma and peripheral layer of plasma In second case blood is represented as casson fluid model.

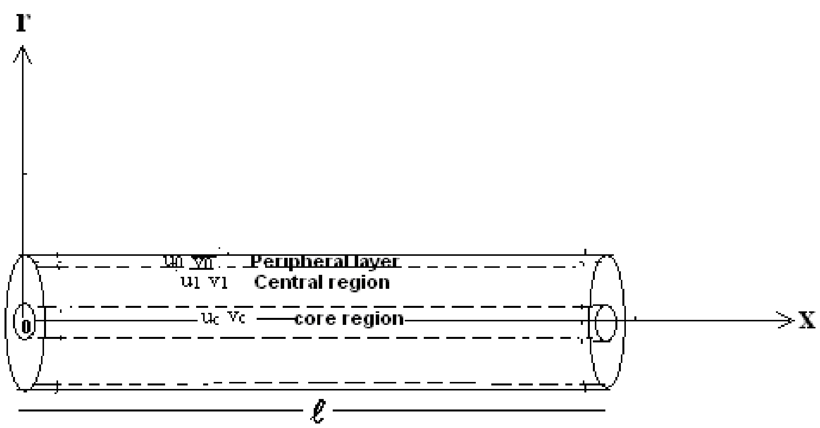

Fig.1 Schematic diagram

A realistic model of the basic function unit should specifically consider the slow viscous flow in narrow capillary therefore the viscosity $\mu_1{}'$ of the blood is considered as depending on the local variation of the concentration as given by the relation $\mu_1' = \mu_0(1+\alpha' c')$

The Concentration c' of the suspended cells in the blood is determined by one dimensional diffusion equation.

$$D'\left[\frac{\partial^2 c'}{\partial r'^2} + \frac{1}{r'}\frac{\partial c'}{\partial r'}\right] + m' = 0$$

Where D' and m' are the diffusion coefficient of undisolved cells and rate of production of cells respectively.

(i) **For Peripheral layer region:** Equation of motion for peripheral layer is given as

$$-\frac{\partial P'}{\partial x'} + \frac{\mu_0}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial u_0'}{\partial r'}\right) = 0 \qquad (1)$$

here μ_0 is the viscosity of plasma, u_0 is axial velocity of plasma in the region

(ii) **For Central region:** Equation of motion for the central region is given as

$$-\frac{\partial P'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\tau_1'\right) = 0 \qquad (2)$$

Where $\mu_1' = \mu_0(1+\alpha' c')$

here μ_1' is the viscosity of central layer which is depend on the viscosity of plasma and concentration of red blood cell, u_1 is axial velocity of blood in the central region

(iii) **For Core layer:** Equation of motion for the core region is given as

$$\frac{\partial u_c'}{\partial r} = 0 \qquad (3)$$

Here u_c' is the velocity of core layer.

Casson Fluid

(i) **For Peripheral layer region:** Equation of motion and equation of continuity for peripheral region are given as

$$-\frac{\partial P'}{\partial x'} + \frac{\mu_0}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial u_0'}{\partial r'}\right) = 0 \qquad (4)$$

$$(R_1 < r < R_2)$$

$$-\frac{\partial u_0'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r' v_0'\right) = 0 \qquad (5)$$

Here u_0 and v_0 are the axial and normal velocities of plasma in peripheral layer and μ_0 is the viscosity of plasma.

(ii) For central region

$$\tau_1^{1/2} = \tau_0^{1/2} + \mu_1^{1/2}\left(-\frac{\partial u_1}{\partial r}\right)^{1/2} \qquad \tau_1 \geq \tau_0 \tag{6}$$

$$\frac{\partial u_1}{\partial r} = 0 \qquad \tau_1 \leq \tau_0 \tag{7}$$

Where τ_1 is stress in central region. τ_0 is constant yield stress in the core region. $\frac{\partial u_1}{\partial r}$ is the rate of strain of the casson fluid. μ_1 denotes casson's viscosity. These relations corresponds to vanishing of velocity gradients in the region where the shear stress τ_1 is less than yield stress τ_0. This implies

Considering the forces on the control volume and equating the shear forces and pressure forces acting on the control volume

$$2\pi(r+dr)\ell(\tau_1+\delta\tau_1) - 2\pi r\ell\tau = -\frac{\partial P}{\partial x}2\pi r\ell\delta r \tag{8}$$

Dividing by δr and taking the limit as $\delta r \to 0$, we get

$$\frac{d}{dr}(r\tau) = -r\frac{\partial P}{\partial r} \tag{9}$$

which on integration leads to

$$\tau_1 = -\frac{\partial P}{\partial r}\frac{r}{2} + \frac{A}{r} \tag{10}$$

The constant A is determined by the condition that τ_1 is finite at $r = 0$ and we get

$$\tau_1 = \frac{r}{2}P \tag{11}$$

Where $P = -\frac{\partial P}{\partial x}$ is the pressure gradient of the flow in capillary.

again putting $r = r_h$, $\tau_1 = \tau_0$ we get

$$\tau_0 = \frac{r_h}{2}P \tag{12}$$

$$-\frac{\partial P'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\tau_1'\right) = 0 \tag{13}$$

$$(r_h < r < R_1)$$

$$-\frac{\partial u_1'}{\partial x'} + \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'v_1'\right) = 0 \tag{14}$$

Where $\tau_1^{1/2} = \tau_0^{1/2} + \mu_1^{1/2}\left(-\frac{\partial u_1}{\partial r}\right)^{1/2}$

(iii) **For Core layer:** Equation of motion for the core region is given as

$$\frac{\partial u_c'}{\partial r} = 0 \tag{15}$$

Here u_c' is the velocity of core layer.

Boundary and Matching conditions: The flow is subject to the following boundary conditions:

(i) No slip condition is assumed at the wall

$u_0' = 0$ at $r' = R_2$

(ii) The velocity and shear stress are continuous at the interphase of plasma (Peripheral layer) and central layer.

$u_0' = u_1'$ at $r' = R_1'$

$\tau_0' = \tau_1'$ at $r' = R_1'$

(iii) Due to symmetry the velocity gradient vanishes along the axis of the tube

$\frac{\partial u_1'}{\partial r'} = \frac{\partial u_c'}{\partial r'} = 0$ at $r' = 0$

(iv) The concentration of red cell in maximum at the centre line and zero at the wall.

$\frac{\partial c'}{\partial r'} = 0$ at $r' = 0$

$c' = 0$ at $r' = R_1'$

Where ρ and u_{b_0} are the density and velocity of the blood and c_0 is some reference concentration of solute.

Solution of the Problem

Three layer model

The expression for the velocities for different layers obtained as the solution of governing equation (1) , (2) and (3) are subject to the boundary condition.

$$u_0 = -\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\left(1 - r^2\right) \tag{16}$$

$$u_1 = \left[-\frac{\mathrm{Re}}{4A}R_1^2\left(1 - \frac{r^2}{R_1^2}\right) - \frac{\alpha M \mathrm{Pe}\,\mathrm{Re}\, R_1^4}{16A^2}\left(1 - \frac{r^4}{R_1^4}\right) - \frac{\mathrm{Re}}{4A}\left(1 - R_1^2\right)\right]\frac{\partial P}{\partial x} \tag{17}$$

$$u_c = \left[-\frac{Re}{4A} R_1^2 \left(1 - \frac{r_h^2}{R_1^2}\right) - \frac{\alpha MPe\,Re\,R_1^4}{16A^2}\left(1 - \frac{r_h^4}{R_1^4}\right) - \frac{Re}{4A}\left(1 - R_1^2\right)\right]\frac{\partial P}{\partial x} \tag{18}$$

The volumetric flow rate is now calculated as

$$Q = Q_0 + Q_1 + Q_c \tag{19}$$

Where $$Q_c = 2\pi \int_0^{r_h} u_c\, r\, dr \;;\quad Q_1 = 2\pi \int_{r_h}^{R_1} u_1\, r\, dr\;;\quad Q_0 = 2\pi \int_{R_1}^{R_2} u_0\, r\, dr \tag{20}$$

Evaluating the integral of (20) we get the flow rate in different regions are given as

$$Q_0 = -2\pi \frac{Re}{4}\frac{\partial P}{\partial x}\left\{\left(\frac{R_2^2 - R_1^2}{2}\right) - \left(\frac{R_2^4 - R_1^4}{4}\right)\right\} \tag{21}$$

$$Q_1 = \left[-\frac{Re}{4A} R_1^2 \left(1 - \frac{r^2}{R_1^2}\right) - \frac{\alpha MPe\,Re\,R_1^4}{16A^2}\left(1 - \frac{r^4}{R_1^4}\right) - \frac{Re}{4A}\left(1 - R_1^2\right)\right]\frac{\partial P}{\partial x} \tag{22}$$

$$Q_c = \left[-\frac{Re}{4A} R_1^2 \left(1 - \frac{r_h^2}{R_1^2}\right) - \frac{\alpha MPe\,Re\,R_1^4}{16A^2}\left(1 - \frac{r_h^4}{R_1^4}\right) - \frac{Re}{4A}\left(1 - R_1^2\right)\right]\frac{r_h^2}{2}\frac{\partial P}{\partial x} \tag{23}$$

Casson Fluid Model

The expression for the velocities in different layers when the central layer is defined as casson fluid. The solution for the velocity in different regions are given by equation no. (24), (25) and (26).

$$u_0 = -\frac{1}{4} Re \frac{\partial P}{\partial x}\left(1 - r^2\right) \tag{24}$$

$$u_1 = \frac{1}{\mu_1}\left[\frac{Re\, r^2}{4}\frac{\partial P}{\partial x} + \tau_0 r - \frac{4}{3}\left(\frac{Re\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2} r^{3/2}\right]$$
$$- \left[\frac{Re}{4}\frac{\partial P}{\partial x}\left(1 - \frac{R_1^2}{R_2^2}\right) + \frac{1}{\mu_1}\left\{\frac{Re}{4}\frac{\partial P}{\partial x}\frac{R_1^2}{4R_2^2} + \tau_0 \frac{R_1}{R_2} - \frac{4}{3}\left(\frac{Re\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2}\left(\frac{R_1}{R_2}\right)^{3/2}\right\}\right] \tag{25}$$

$$u_c = \frac{1}{\mu_1}\left[\frac{\mathrm{Re}\, r_h^{\,2}}{4}\frac{\partial P}{\partial x} + \tau_0 r_h - \frac{4}{3}\left(\frac{\mathrm{Re}\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2} r_h^{\,3/2}\right]$$
$$- \left[\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\left(1-\frac{R_1^{\,2}}{R_2^{\,2}}\right) + \frac{1}{\mu_1}\left\{\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\frac{R_1^{\,2}}{4R_2^{\,2}} + \tau_0\frac{R_1}{R_2} - \frac{4}{3}\left(\frac{\mathrm{Re}\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2}\left(\frac{R_1}{R_2}\right)^{3/2}\right\}\right] \tag{26}$$

The volumetric rate for this case is calculated as

$$Q = Q_0 + Q_1 + Q_c$$

Where $\quad Q_c = 2\pi\int_0^{r_h} u_c\, r\, dr \;; \quad Q_1 = 2\pi\int_{r_h}^{R_1} u_1\, r\, dr\,; \quad Q_0 = 2\pi\int_{R_1}^{R_2} u_0\, r\, dr \qquad (27)$

Solving the integral of equation no. (27) volumetric flow rate for the different regions are given as

$$Q_0 = -2\pi\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\left\{\left(\frac{R_2^{\,2}-R_1^{\,2}}{2}\right) - \left(\frac{R_2^{\,4}-R_1^{\,4}}{4}\right)\right\} \tag{28}$$

$$Q_1 = \frac{1}{\mu_1}\left[\frac{\mathrm{Re}}{4}\left(\frac{R_1^{\,4}-r_h^{\,4}}{4}\right)\frac{\partial P}{\partial x} + \tau_0\left(\frac{R_1^{\,3}-r_h^{\,3}}{3}\right) - \frac{8}{21}\left(\frac{\mathrm{Re}\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2}\left(\frac{R_1^{\,7/2}-r_h^{\,7/2}}{4}\right)\right]$$
$$- \left[\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\left(1-\frac{R_1^{\,2}}{R_2^{\,2}}\right) + \frac{1}{\mu_1}\left\{\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\frac{R_1^{\,2}}{4R_2^{\,2}} + \tau_0\frac{R_1}{R_2} - \frac{4}{3}\left(\frac{\mathrm{Re}\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2}\left(\frac{R_1}{R_2}\right)^{3/2}\right\}\left(\frac{R_1^{\,2}-r_h^{\,2}}{2}\right)\right] \tag{29}$$

$$Q_c = \frac{1}{\mu_1}\left[\frac{\mathrm{Re}\, r_h^{\,4}}{8}\frac{\partial P}{\partial x} + \tau_0\frac{r_h^{\,3}}{2} - \frac{4}{6}\left(\frac{\mathrm{Re}\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2} r_h^{\,5/2}\right]$$
$$- \left[\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\frac{r_h^{\,2}}{2}\left(1-\frac{R_1^{\,2}}{R_2^{\,2}}\right) + \frac{1}{\mu_1}\left\{\frac{\mathrm{Re}}{4}\frac{\partial P}{\partial x}\frac{R_1^{\,2}}{4R_2^{\,2}} + \tau_0\frac{R_1}{R_2} - \frac{4}{3}\left(\frac{\mathrm{Re}\,\tau_0}{2}\frac{\partial P}{\partial x}\right)^{1/2}\left(\frac{R_1}{R_2}\right)^{3/2}\right\}\frac{r_h^{\,2}}{2}\right] \tag{30}$$

Results & Discussion

The graphs between flow rate and pressure gradient has been shown in the figures 2 & 3. It is clear from the figure that pattern is same as those for model of V.P. Srivastava and casson fluid model. It has been concluded that present model suitable describes blood flow in small vessels at low concentration of red cells. The results of the analysis deviate from the experimental works with increasing diameter of blood vessels and also with increasing hematocrit. The reason behined this is the empirical formula used for the viscosity is based on the Einstein's theory of particulate suspension and is therefore applicable only for low particle concentration.

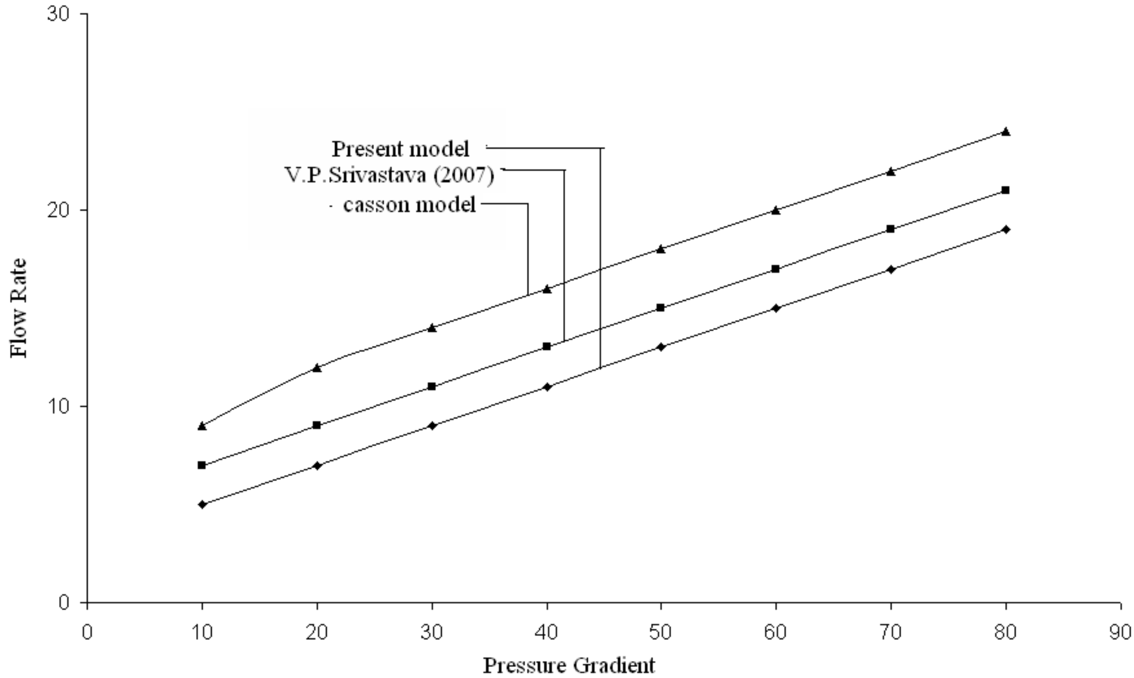

Fig.2 variation of flow rate with Pressure gradient for concentration coefficient

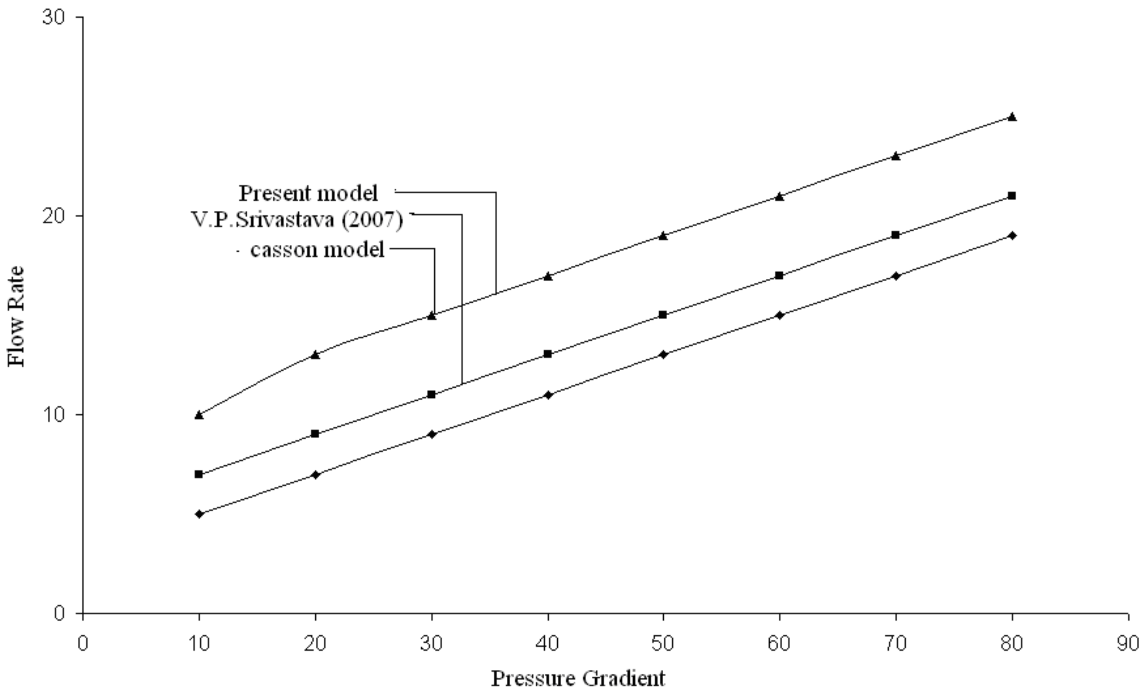

Fig. 3 variation of flow rate with Pressure gradient for concentration coefficient 1.5

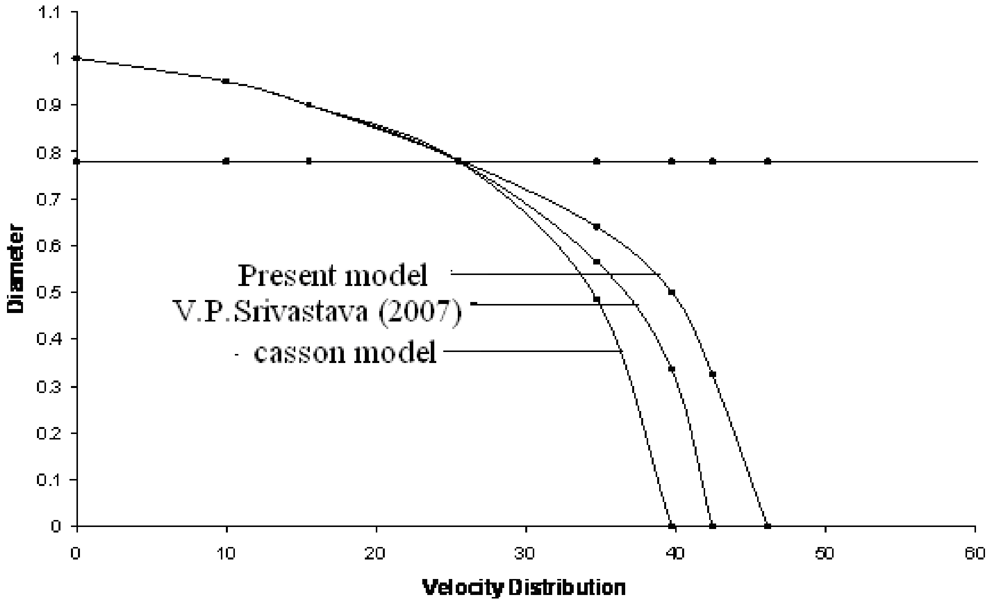

Fig. 4 velocity distribution with diameter for concentration coefficient 0.5

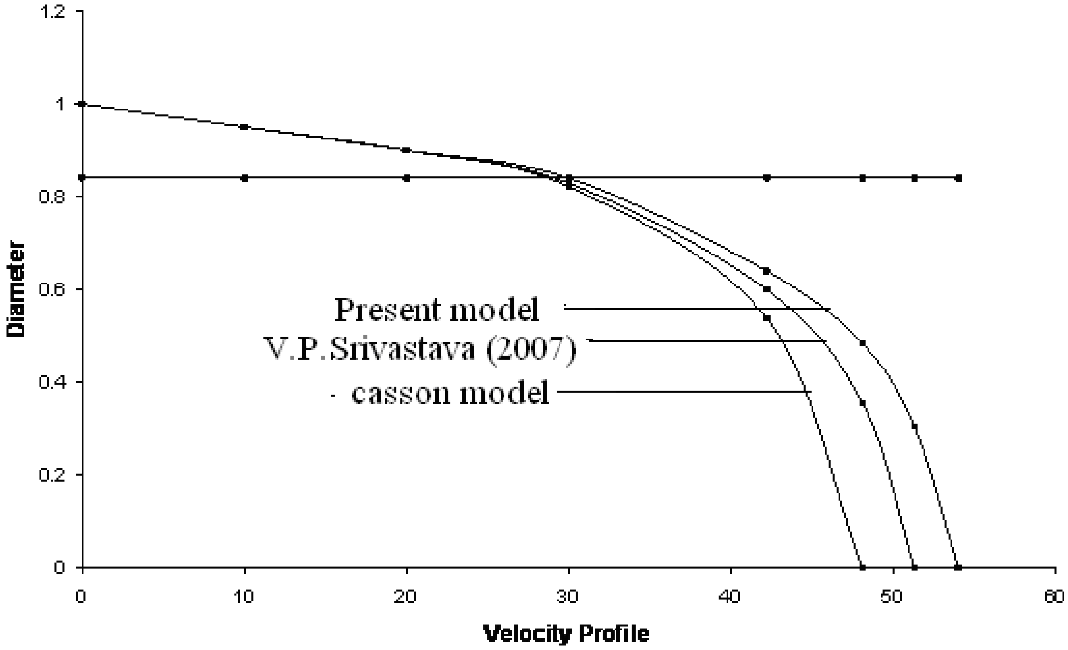

Fig. 5 velocity distribution with diameter for concentration coefficient 1.5

The axial velocity Profile for the present model has been compared with the results of model of V. P. Srivastava (2007) (using erythrocytes-plasma suspension to represent blood in the core region) and casson fluid model. These graphs are shown in figure 4&5. We have observed that velocity at the tube axis is more than plasma velocity in peripheral layer. The difference in the magnitude decreases with increasing radial coordinate. The plasma velocity coincides with the blood velocity at the interface. Our results are similar to those obtained by V. P. Srivastava.

Conclusions

A three layer model for the blood flow and casson fluid model has been considered and compared with the particle fluid mixture model of V.P. Srivastava (2007). The velocity profile and flow rate has been obtained and discussed through graph**.**
It has been concluded that blood in narrow tube can be modeled by a three layer model of fluid as the result for flow rate and velocity profile shows similar variation of those obtained by V. P. Srivastava(2007) and casson fluid model.

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